The first clause of the axiom guarantees that an arbitrary individual cannot play the role of the empty set 0. Justifying reasons for the restrictions on the variable **v** are given in the next section.

In principle all of the axioms and theorems of set theory that we state in the following pages can be written as primitive formulas of the object language — indeed, our official object language shall consist of these primitive formulas. For working purposes it will be useful and convenient to introduce by definition considerable additional notation. We shall in practice apply the axiom schema of separation to formulas which are not written solely in primitive notation; but since at any point in our development only a finite number of definitions will have preceded, such a formula can be replaced by a primitive formula by a finite number of substitutions.

Regarding definitions, then, our viewpoint is that they are informally admitted if clear recipes are given for eliminating new symbols from any context. We thus require that a formula of the object language which introduces a new symbol must satisfy the following:

> CRITERION OF ELIMINABILITY. *A formula* P *introducing a new symbol satisfies the criterion of eliminability if and only if: whenever* Q_1 *is a formula in which the new symbol occurs, then there is a primitive formula* Q_2 *such that* $\mathsf{P} \rightarrow (\mathsf{Q}_1 \leftrightarrow \mathsf{Q}_2)$ *is derivable from the axioms.*

Notice that we have stated this criterion without giving an exact definition of *formula* (as opposed to *primitive formula*). Such a definition is straightforward if we list all the defined symbols introduced in this book, and then proceed in terms of this list as we did before with the primitive notation. This tedious task we shall not perform, but we do want to mention a second criterion we expect our definitions to satisfy, namely, our definitions must not be creative.

> CRITERION OF NON-CREATIVITY. *A formula* P *introducing a new symbol satisfies the criterion of non-creativity if and only if: there is no primitive formula* Q *such that* $\mathsf{P} \rightarrow \mathsf{Q}$ *is derivable from the axioms but* Q *is not.*

In other words, a definition should not function as a creative axiom permitting derivation of some previously unprovable formula in which only primitive notation occurs.

The classical problem of the theory of definition for any exactly stated mathematical theory is to provide rules of definition whose satisfaction entails satisfaction of the two criteria just stated. We may restrict ourselves here to rules for defining operation symbols. Slight modifications yield appropriate rules for defining relation symbols and individual constants.* In these rules we refer to *preceding definitions*, which implies that the definitions are given in a fixed sequence and not simultaneously;

*Individual constants may in fact be treated as operation symbols of degree zero.

Expressions of the object language are finite sequences of the five classes of symbols of the language. Certain of these expressions, simply because of their structure, are called *primitive formulas* of the object language. We now define such formulas so that merely by looking at the form of an expression we can automatically decide in a finite number of steps whether or not it is a primitive formula. Although this definition is purely syntactical or structural, it is just the expressions satisfying it which have a clear intuitive meaning. An expression like '$(\rightarrow \in x$' is not a primitive formula and has no intuitive meaning.

We first define primitive *atomic* formulas.

> *A primitive atomic formula is an expression of the form* $(\mathbf{v} \in \mathbf{w})$, *or of the form* $(\mathbf{v} = \mathbf{w})$, *where* $\mathbf{v}$ *and* $\mathbf{w}$ *are either general variables or the constant* '0'.*

Thus '$x \in y$' and '$z = 0$' are primitive atomic formulas.

We may now give what is usually called a recursive definition of *primitive formulas:*

(a) *Every primitive atomic formula is a primitive formula;*
(b) *If* $\mathbf{P}$ *is a primitive formula, then* $-\mathbf{P}$ *is a primitive formula;*
(c) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are primitive formulas, then* $(\mathbf{P} \,\&\, \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \rightarrow \mathbf{Q})$, *and* $(\mathbf{P} \leftrightarrow \mathbf{Q})$ *are primitive formulas;*
(d) *If* $\mathbf{P}$ *is a primitive formula and* $\mathbf{v}$ *is any general variable then* $(\forall \mathbf{v})\mathbf{P}$, $(\exists \mathbf{v})\mathbf{P}$ *and* $(\mathrm{E}!\mathbf{v})\mathbf{P}$ *are primitive formulas;*
(e) *No expression of the object language is a primitive formula unless its being so follows from rules* (a) – (d).

The following are examples of primitive formulas of the object language which are not atomic: '$(\exists x)(\forall y) - (y \in x)$', '$x \in y \rightarrow y \in z$', '$(\mathrm{E}!\, z)(0 = z)$'. In terms of this definition, an exact formulation of the axiom schema of separation is then:

Any primitive formula of the object language of the form

$$(\exists \mathbf{v})((\exists \mathbf{w}_1)(\mathbf{w}_1 \in \mathbf{v} \vee \mathbf{v} = 0) \,\&\, (\forall \mathbf{w})(\mathbf{w} \in \mathbf{v} \leftrightarrow \mathbf{w} \in \mathbf{u} \,\&\, \varphi))$$

is an axiom, provided the variable $\mathbf{v}$ *is distinct from* $\mathbf{u}$ *and* $\mathbf{w}_1$ *and is not free in the primitive formula* φ.

*In this definition, as elsewhere, we use the boldface letters '**u**', '**v**', '**w**', '$\mathbf{u}_1$', '$\mathbf{v}_1$', '$\mathbf{w}_1$', . . . as metamathematical variables which take as values variables 'x', 'y', 'z', . . . or the constant '0' of the object language. And we use boldface letters, '**P**', '**Q**', . . . , as well as Greek letters 'φ' and 'Ψ', as metamathematical variables which take as values formulas of the object language. The conventions about use and mention followed here, which are probably obvious, are that (i) the constants '$\in$' and '$=$', the sentential connectives, the quantifier symbols, and the left and right parentheses are used as names of themselves, and (ii) juxtaposition of names of expressions denotes a binary operation on expressions which yields new expressions (for example, '$x \in y$' & '$y \in z$' = '$x \in y$ & $y \in z$'). For a more detailed discussion of these conventions, see Chapter 6 of Suppes [1957].

The first clause of the axiom guarantees that an arbitrary individual cannot play the role of the empty set 0. Justifying reasons for the restrictions on the variable **v** are given in the next section.

In principle all of the axioms and theorems of set theory that we state in the following pages can be written as primitive formulas of the object language — indeed, our official object language shall consist of these primitive formulas. For working purposes it will be useful and convenient to introduce by definition considerable additional notation. We shall in practice apply the axiom schema of separation to formulas which are not written solely in primitive notation; but since at any point in our development only a finite number of definitions will have preceded, such a formula can be replaced by a primitive formula by a finite number of substitutions.

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In other words, a definition should not function as a creative axiom permitting derivation of some previously unprovable formula in which only primitive notation occurs.

The classical problem of the theory of definition for any exactly stated mathematical theory is to provide rules of definition whose satisfaction entails satisfaction of the two criteria just stated. We may restrict ourselves here to rules for defining operation symbols. Slight modifications yield appropriate rules for defining relation symbols and individual constants.* In these rules we refer to *preceding definitions*, which implies that the definitions are given in a fixed sequence and not simultaneously;

*Individual constants may in fact be treated as operation symbols of degree zero.

this approach permits use of defined symbols in the definitions of new symbols.*

Proper definitions of operation symbols may be either equivalences or identities. We begin with the former.

An equivalence P *introducing a new n-place operation symbol* O *is a proper definition if and only if* P *is of the form*

$$\mathsf{O}(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{w} \leftrightarrow \mathsf{Q}$$

and the following restrictions are satisfied: (i) $\mathsf{v}_1, \ldots, \mathsf{v}_n, \mathsf{w}$ *are distinct variables,* (ii) Q *has no free variables other than* $\mathsf{v}_1, \ldots, \mathsf{v}_n, \mathsf{w}$, (iii) Q *is a formula in which the only non-logical constants are the primitive or previously defined symbols of set theory, and* (iv) *the formula* $(\mathrm{E}!\, w)\mathsf{Q}$ *is derivable from the axioms and preceding definitions.*

Regarding the phrase 'non-logical constants' in (iii), the only logical constants are those introduced in §1.2; all other constants are non-logical. Justification of the various restrictions is easily given. Here we shall emphasize only the importance of (iv). Consider the following definition in elementary arithmetic of the pseudo-operation $\star$.

(1) $$x \star y = z \leftrightarrow x < z \;\&\; y < z.$$

Clearly it is false that

$$(\mathrm{E}!z)\ (x < z \;\&\; y < z).$$

Thus (1) violates (iv), and we want to show that violation of it will lead to a contradiction. Now since $1 < 3$, $2 < 3$, $1 < 4$, and $2 < 4$, we immediately infer from (1):

$$1 \star 2 = 3$$

and

$$1 \star 2 = 4$$

whence

$$4 = 3,$$

which is absurd. In ordinary mathematical language the point of (iv) is to require that performing an operation shall always yield a unique object.

For a definition which is an identity we have the following rule.

An identity P *introducing a new n-place operation symbol* O *is a proper definition if and only if* P *is of the form*

$$\mathsf{O}(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{t}$$

*The rules to be given and related matters are discussed in more detail in Chapter 8 of Suppes [1957].

and the following restrictions are satisfied: (i) $\mathsf{v}_1, \ldots, \mathsf{v}_n$ *are distinct variables,* (ii) *the term* t *has no free variables other than* $\mathsf{v}_1, \ldots, \mathsf{v}_n$, *and* (iii) *the only non-logical constants in the term* t *are primitive symbols and previously defined symbols of set theory.*

An example of a definition by means of an identity in arithmetic is the definition of subtraction in terms of addition and the negative operation.

$$x - y = x + (-y).$$

It is straightforward to prove that definitions satisfying either of these rules just given, or the analogous ones for relation symbols and individual constants, satisfy the criteria of eliminability and non-creativity.

Unfortunately many of the definitions common in mathematics and many of the definitions to be introduced in the sequel do not satisfy the criterion of eliminability; and thus most of the definitions of operation symbols do not satisfy one of the two rules introduced. The reason for this failure may be simply stated: the definitions are often conditional in form. A typical instance of a conditional definition in arithmetic is provided by a definition of division, for which the problem of division by zero arises.

$$(1) \qquad y \neq 0 \rightarrow (x/y = z \leftrightarrow x = y \cdot z).$$

Using (1) as the definition of the operation symbol for division, we cannot eliminate the symbol from contexts like:

$$1/0 \neq 2.$$

On the other hand, we can use (1) to eliminate division in all "interesting" cases, that is, all those which satisfy the hypothesis of (1). Moreover, it is not difficult to modify the two rules given in such a way that conditional definitions satisfying them satisfy the criterion of non-creativity. In fact, the appropriate modifications of the rule for equivalences which define operation symbols are embodied in the following.

An implication P *introducing a new operation symbol* O *is a conditional definition if and only if* P *is of the form*

$$\mathsf{Q} \rightarrow [\mathsf{O}(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{w} \leftrightarrow \mathsf{R}]$$

and the following restrictions are satisfied: (i) *the variable* w *is not free in* Q, (ii) *the variables* $\mathsf{v}_1, \ldots, \mathsf{v}_n$, w *are distinct,* (iii) R *has no free variables other than* $\mathsf{v}_1, \ldots, \mathsf{v}_n$, w, (iv) Q *and* R *are formulas in which the only non-logical constants are the primitive symbols and previously defined symbols of set theory, and* (v) *the formula* $\mathsf{Q} \rightarrow (\mathrm{E}!\mathsf{w})\mathsf{R}$ *is derivable from the axioms and preceding definitions.*

To convert a conditional definition of an operation symbol into a proper definition satisfying the criterion of eliminability is a routine matter once