Set Theory week 4 Due Monday Feb. 8^{th}

Wednesday: The role of well-ordering in recursion

(Prove the following claims)

Definition: Let $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ be linear orders. A function $\varphi: A \to B$ is called order preserving if $\forall a_1, a_2 \in A$, $a_1 < a_2 \iff \varphi(a_1) < \varphi(a_2)$. Note that this forces φ to be injective. Thus, we may also say that φ is an embedding of A into B

Definition: We denote the linearly ordered set $\langle \omega, <^* \rangle$ by ω^*, where

$$m <^* n \iff n \in m$$

1) Claim: Let $\langle A, < \rangle$ be a linear order. Then A is a well-order if and only if there does not exist an embedding of ω^* into A.

Definition: Let $\langle W, <_W \rangle$ be a well-ordered set and let $\{B_w\}_{w \in W}$ be a family of linearly ordered sets indexed by W. Denote by $<_{B_w}$ the order on B_w. The lexicographical order of $\prod_{w \in W} B_w$ is given by $\tau <_{lex} \sigma$ if and only if $\tau(\omega_0) <_{B_{\omega_0}} \sigma(\omega_0)$, where ω_0 is $<_W$-least in $\{\omega \in W / \tau(\omega) \neq \sigma(\omega)\}$

2) Claim: Let $\{A_n\}_{n \in \omega}$ be a family of linearly ordered sets such that $|A_n| \geq 2$, $\forall n \in \omega$. Then the lexicographical order on $\prod_{n \in \omega} A_n$ is not a well-order.

Definition: Let $\langle A, < \rangle$ be a linearly ordered set. A set $I \subseteq A$ is called downward-closed (or sometimes an initial segment) if, if $y < x$ and $x \in I$ implies $y \in I$. It is called a proper initial segment if $I \subsetneq A$. Also sometimes denoted as $I \sqsubseteq A$ and $I \sqsubsetneq A$. The set $A[a] = \{x \mid x < a\}$ is called the initial segment given by a.

$\blacksquare$

Definition: Let $\langle A, < \rangle$ be a linearly ordered set, let $f: A \to X$ be a function and let $a \in A$. We use $f\restriction_a$ to denote f restricted to $A[a]$.

$\blacksquare$

Consider the following set-up that generalizes the hypotheses of the recursion theorems;

[Let $\langle A, < \rangle$ be a linearly ordered set. Let $G: V \to V$ be an operation (class function). We will call a function $f: A \to X$ determined by G if

$$(*) \quad \forall a \in A, \; f(a) = G(f\restriction_a)$$
]

Given two linear orders A and B, we can form their sum $A+B$ via disjoint union just like ordinals.

* 3) Consider the ordering $\omega + \omega^*$. Give examples of operations $G: V \to V$ such that

(i) there exists an $f: \omega + \omega^* \to 2$ determined by G but it is not unique.

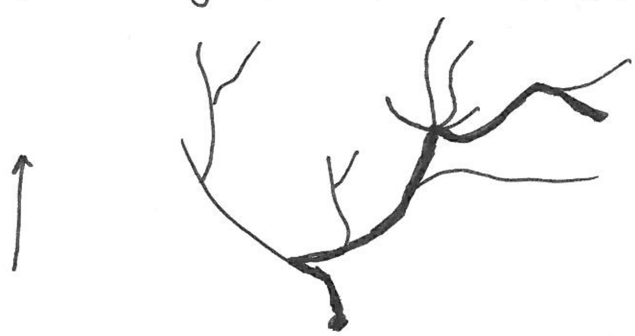
(ii) there does not exist an $f: \omega + \omega^* \to 2$ determined by G

*4) <u>Claim</u>: Let $\langle A, < \rangle$ be given as in [] on pg. 2. If $\exists I \subsetneq A$ and an order isomorphism $\psi : A \xrightarrow{\sim} I$, then there is a G and X as in [] such that there exists a non-unique $f: A \to X$ determined by G.

It should now be clear that the well-ordering structure is <u>necessary</u> for recursion!

Eliminating the possibility of infinite descending sequences prevents <u>ill-founded</u> recursion.

To ignite the imagination, suppose you have an infinitely branching tree with infinite branches.



(bolded section represents a branch)

Then so long as the branches are well-orderings ~~and~~ ~~[illegible]~~ we can make branching recursive transfinite computations.