Set Theory Week 4 Due ~~Monday~~ Friday Feb. 5th

Monday: Peano Arithmetic

(Prove the following claims)

You may take as given that $+: \omega \times \omega \to \omega$ is commutative.

1) **Claim:** Addition is associative. That is, $\forall m, n, k \in \omega$, $(m+n)+k = m+(n+k)$.

2) **Claim:** For any $m, n \in \omega$, $m \leq n$, $\exists! \, k \in \omega$ such that $n = m+k$. Thus we may define $n - m = k$.

3) **Claim:** Let $m, n, k \in \omega$, $k > 0$. Then $m < n \iff m \cdot k < n \cdot k$.

You may now assume all basic properties of addition and multiplication on natural numbers.

Definition: We may define exponentiation of natural numbers using ω-recursion, as follows:

Let $m > 0$.

$$m^0 = 1$$
$$m^{n+1} = m^n \cdot m$$

4) **Claim:** $\forall m, n, k \in \omega$, $m > 0$.

(i) $m^n \cdot m^k = m^{n+k}$

(ii) $(m^n)^k = m^{n \cdot k}$

Now we may be quite confident that the operations conform to the structure we intend to represent.

I should not say it is "trivial" though the proofs are mechanical. The magic is in the correct choice of axioms and definition. Just like the "glorious" (generalized) Stoke's Thm, which is highly non-trivial.