Week 3 Wednesday

Set Theory

Due Monday Feb 1st

Having fun with cardinalities. (prove the following claims)

1)

Note. It is the case that $|[0,1]| =_c |(0,1)| =_c |\mathbb{R}|$. The first equality can be shown easily by producing injective functions in both directions. The function $f(x) = \dfrac{x - 1/2}{x(x-1)}$ on the interval $(0,1)$ witnesses the second equality.

Claim. $|2^\omega| =_c |\mathcal{P}(\omega)|$

Claim. $|2^\omega| =_c |\mathbb{R}|$

***2)**

Remark. Recall that a set S is said to be <u>Dedekind infinite</u> if there exists an injection from S into a proper subset of S.

Claim. Infinite implies Dedekind infinite.

***3)**

Definition. A number $a \in \mathbb{R}$ is called <u>algebraic</u> if it is the root of some polynomial $p(x) = q_n x^n + \ldots + q_1 x + q_0$ with coefficients $q_i \in \mathbb{Q}$. A number $a \in \mathbb{R}$ is <u>transcendental</u> if it is not algebraic.

Claim. The set of all algebraic numbers is countable, from which we may conclude that transcendental numbers exist.

****4)**

Claim. Let $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous }\}$. Then $|C_0(\mathbb{R})| =_c |\mathbb{R}|$.

<u>Hint:</u> You will likely find it useful to prove and use the following fact. The countable product of sets of size $|\mathbb{R}|$ has size $|\mathbb{R}|$, for which it suffices to show that $|(2^\omega)^\omega| =_c |2^\omega|$.