Week 3 Friday

Set Theory

Due Monday Feb 1st

Applying the ω-recursion theorem.

(prove the following claims)

*__1)__

Definition. Let X be a set. Consider any family of functions

$$\mathcal{F} \subseteq \{f \mid f : X \to X\}$$

Let $Y \subseteq X$. We say that $\underline{Y \text{ is closed under } \mathcal{F}}$ if $\forall f \in \mathcal{F}$ and $\forall y \in Y$ we have that $f(y) \in Y$.

Claim. Let X be a set. Let $\mathcal{F}$ be a countable family of functions of the form $f : X \to X$. Then $\forall Y \subseteq X$, there exists a set $Y_{\mathcal{F}}$ such that $Y \subseteq Y_{\mathcal{F}}$, $Y_{\mathcal{F}}$ is closed under $\mathcal{F}$ and $Y_{\mathcal{F}}$ is minimal with respect to set containment (i.e. $\forall Z \subseteq X$ such that $Y \subset Z$ and such that Z is closed under $\mathcal{F}$, it is the case that $Y_{\mathcal{F}} \subseteq Z$).

**__2)__

Definition. A linear order $\langle A, \prec \rangle$ is called <u>without endpoints</u> if $\forall a \in A$, $\exists b, c \in A$ such that $b < a$ and $a < c$. It is called <u>dense</u> if $\forall a, b \in A$ such that $a < b$, $\exists c \in A$ such that $a < c < b$.

Note that examples of dense linear orders without endpoints include $\mathbb{Q}$ and $\mathbb{R}$ with their usual ordering. Also note that, if there exists at least one element, a linear order without endpoints is necessarily infinite. Similarly, if there are at least 2 elements, then a dense linear order is necessarily infinite. For further intuition, notice that you can think of $\mathbb{Q}$ as the "density closure" of $\mathbb{Z}$, in the sense that $\mathbb{Z}$ retains its usual ordering inside $\mathbb{Q}$ and $\mathbb{Q}$ is minimal in the sense that it adds no more order structure than is necessary to extend $\mathbb{Z}$ to a dense linear order. Also notice that $\mathbb{Q}$ is order isomorphic to any open interval inside $\mathbb{Q}$, such as $(0, 1) \cap \mathbb{Q}$ (any of these observations must be proved in your write-up if you choose to use them).

Theorem. Let $\langle A, \prec_A \rangle$ and $\langle B, \prec_B \rangle$ be countably infinite, dense linear orders without endpoints. Then $A \cong B$ (i.e. there exists a bijection $\psi : A \to B$ such that $\forall a_1, a_2 \in A$, $a_1 < a_2$ if and only if $\psi(a_1) < \psi(a_2)$).