Week 2 Monday

Set Theory

Due Wednesday Jan 20th

This homework tests understanding of the process of translating the usual mathematical presentation into a first order theory. You need only use formal first order formulas where it is clear that you should do so.

1) The Theory of Gadgets:

(i) Turn the following description of a class of mathematical objects into a first order theory. This means picking a language/signature (i.e. all the relation, function and constant symbols in the language), then choosing a set of well-formed formulas to create your theory.

Definition. We say that X, $\triangle$, $\square$, and $\bigcirc$ form a gadget if the following conditions are satisfied. There is some $\gamma \in X$ such that, for any $x, y \in X$, $\triangle(\gamma, x, y) = \triangle(x, \gamma, y) = \triangle(x, y, \gamma) = \gamma$. A triple $< x, y, z > \in X \times X \times X$ is $\square(x, y, z)$ if and only if $\triangle(x, y, z) = \triangle(y, \triangle(y, x, z), z)$. An element $x \in X$ is $\bigcirc(x)$ if and only if, for any $y, z \in X$, it is true that $\square(x, y, z)$.

(ii) Is it true that, for any gadget X, $\exists x \bigcirc(x)$? Give a counter example or a proof, accordingly.

*(iii) Complete the following definition of gadget isomorphism (remember, an isomorphism must respect both the relation and function symbols of the language):

Definition. Let $\mathcal{M} =< X, \triangle^{\mathcal{M}}, \square^{\mathcal{M}}, \bigcirc^{\mathcal{M}} >$ and $\mathcal{N} =< X, \triangle^{\mathcal{N}}, \square^{\mathcal{N}}, \bigcirc^{\mathcal{N}} >$ be gadgets. A function $\psi : X \to Y$ is called a gadget isomorphism if it is one-to-one and onto (i.e. a bijection) and satisfies the following 3 conditions:

[You give the three conditions]

The following vector space axioms are given for reference in **2)**.

Definition. Let $\mathbb{F}$ be a field. A set V along with binary operations $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ is called a *vector space* if it satisfies the following conditions:

i. $\forall x, y \in V,\ x + y = y + x$

ii. $\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$

iii. $\exists 0 \in V,\ \forall x \in V,\ x + 0 = x$

iv. $\forall x \in V\ \exists y \in V,\ x + y = 0$

v. $\forall x \in V,\ 1 \cdot x = x$

vi. $\forall a, b \in \mathbb{F}\ \forall x \in V,\ (ab) \cdot x = a \cdot (b \cdot x)$

vii. $\forall a \in \mathbb{F}\ \forall x, y \in V,\ a \cdot (x + y) = a \cdot x + a \cdot y$

viii. $\forall a, b \in \mathbb{F}\ \forall x \in V,\ (a + b) \cdot x = a \cdot x + b \cdot x$

*__2)__ The Theory the Vector Spaces:

Let $\mathbb{F}$ be a field. Create a first order language $\widehat{\mathcal{L}} = \mathcal{L} \cup \mathcal{L}^*$ (i.e. the core first order logic symbols with an additional set of relation and function symbols $\mathcal{L}^*$ chosen by you) and a theory T_{VS} in the language $\widehat{\mathcal{L}}$ whose models are exactly the vector spaces over $\mathbb{F}$.

Hint: You'll want to make the universe (of discourse) of each model (i.e. the set associated with the model) the vectors in your vector space. This makes each formula that looks like $\forall a \in \mathbb{F}\ \phi(a)$ not directly translatable into our first order language (this is because the $\forall$ is interpreted as ranging over the vectors, not the scalars). However, we can replace each instance of $\forall a \in \mathbb{F}\ \phi(a)$ by a collection of formulas $\{\phi(a)\}_{a \in \mathbb{F}}$. Now, for each $\phi(a)$ to be a legitimate formula in a first order language, we need only replace the instance of a with a language symbol that fulfills the same functionality as a.