Math 89
Winter 2008
Written Homework: Friday, January 18, 2008

In class we talked about developing the syntax of first-order logic inside the universe of set theory (or from the axioms of set theory). We choose sets to represent symbols of the language, define expressions to be finite sequences of symbols, terms and formulas to be expressions with certain properties, deductions to be finite sequences of expressions with certain properties, and so forth. We show that we can talk meaningfully about syntactic concepts such as a variable occurring free in a formula. (Formally, we show that there is— in the "real world"– a formula φ of the language of set theory such that, for any language $\mathcal{L}$, formula p of $\mathcal{L}$— in the sense that we define formula inside the universe of sets—, and variable x,

$$\varphi(\mathcal{L}, p, x)$$

expresses the property "p is a formula of $\mathcal{L}$ in which the variable x occurs free.)

Now we want to move on to semantics. We will restrict our attention to the language $\mathcal{L}$ with a single binary predicate symbol E. We define a *structure* for $\mathcal{L}$ to be a pair $\mathfrak{A} = (A, R)$, where A is a nonempty set and R is a binary relation on A. We define a *variable assignment* for $\mathfrak{A}$ to be a function from the set of variables to A. We want to be able to talk meaningfully about when a formula p of $\mathcal{L}$ is *satisfied* by the structure $\mathfrak{A}$ with the variable assignment v. We are following Enderton here:

Problem:

Let $\mathfrak{A} = (A, R)$ be a structure for $\mathcal{L}$ and $\mathcal{V}$ be the set of all variable assignments for $\mathfrak{A}$. Let $\mathfrak{F}$ be the set of all formulas of $\mathcal{L}$. Let $F = 0$ and $T = 1$. Show there is a unique function

$$Sat : (\mathfrak{F} \times \mathcal{V}) \to \{T, F\}$$

such that, for all formulas, variables, and variable assignments:

$$Sat((x = y), v) = T \iff v(x) = v(y)$$

$$Sat((x\, E\, y), v) = T \iff (v(x),\, v(y)) \in R$$

$$Sat(\not p, v) = T \iff Sat(p, v) = F$$

$$Sat(p \wedge q, v) = T \iff Sat(p, v) = Sat(q, v) = T$$

and finally, $Sat((\forall x\ p), v) = T$ if and only if, for every variable assignment w that agrees with v on every variable except perhaps x, we have $Sat(p, w) = T$.