Dartmouth College

Mathematics 81

This is part of the problem set is due on Wednesday, 26 January.

1. Let F be a field of characteristic 0, and let m and n be integers with $\sqrt{m} \notin F$, $\sqrt{n} \notin F$, and $\sqrt{mn} \notin F$.
   (a) Show that $[F(\sqrt{m}, \sqrt{n}) : F] = 4$.
   (b) Show by example that the above proposition is false if we only assume that $\sqrt{m} \notin F$ and $\sqrt{n} \notin F$.

2. Let $m_1, m_2, \ldots, m_t$ be integers none of which are squares in $\mathbb{Z}$.
   (a) Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] \leq 2^t$, and give an example to show that the inequality can be strict.
   (b) Now assume that the integers are square-free and relatively prime in pairs. Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] = 2^t$. Hint: Induction on t and problem 1 may be of use.

3. Consider the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$. Determine a basis for this extension. Hint: Rather than trying to prove directly that the set you write down is linearly independent, give an argument, based on what we have done in class, which proves your set is a basis. Your argument should extend easily to $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}$ for which you should simply state (with confidence) a basis.

4. Determine the degree of the extension $\mathbb{Q}(i, \sqrt{3}, e^{2\pi i/3})/\mathbb{Q}$, and write down three intermediate fields K, (i.e, with $\mathbb{Q} \subsetneq K \subsetneq \mathbb{Q}(i, \sqrt{3}, e^{2\pi i/3})$).