Dartmouth College

Mathematics 81

This problem is part of the assignment due on Wednesday, 14 January.

Let p be a prime in $\mathbb{Z}$, and consider two multiplicative subsets of $\mathbb{Z}$: $S = \mathbb{Z} \setminus p\mathbb{Z}$, and $T = \{1, p, p^2, \dots\}$. The localization $S^{-1}\mathbb{Z}$ is denoted $\mathbb{Z}_{(p)}$ and called the localization of $\mathbb{Z}$ at the prime p.

1. Characterize $\mathbb{Z}_{(p)}$ as a subset of $\mathbb{Q}$, that is $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid \text{put your conditions here}\}$, and characterize the unit group $\mathbb{Z}_{(p)}^{\times}$.
2. Characterize $T^{-1}\mathbb{Z}$ as a subset of $\mathbb{Q}$, and characterize its unit group.
3. The ring $\mathbb{Z}[\frac{1}{p}]$ is the homomorphic image of the polynomial ring $\mathbb{Z}[x]$ under the evaluation homomorphism which takes $x \mapsto 1/p$. Show that $T^{-1}\mathbb{Z} = \mathbb{Z}[\frac{1}{p}]$.
4. Show that for any prime $q \neq p$, $\mathbb{Z}[\frac{1}{p}] \subset \mathbb{Z}_{(q)}$.
5. Finally show that $\mathbb{Z}[\frac{1}{p}] = \bigcap_{q \neq p} \mathbb{Z}_{(q)}$; the intersection is over all primes q of $\mathbb{Z}$ except p.