Math 75 – Homework #3

posted April 11, 2008; due Monday, April 14, 2008

Exercises (Additional problems may be added to this assignment.)

1. Suppose F is a field with characteristic p; that is, p is a prime number and $p1 = 0$. Show that for each $\beta \in F$ we have $p\beta = 0$.
2. Give a careful proof of equation (2) from the April 9 lecture notes.
3. Give a careful proof of equation (4) from the April 9 lecture notes.
4. Let $F = \mathbf{Z}/(2)$. Give a complete factorization into irreducible polynomials over F of $x^{16} - x$.
5. Suppose F is a finite field with q elements and let $f(x) \in F[x]$ have degree d. Show that f is irreducible if and only if the monic gcd of $f(x)$ and $x^{q^j} - x$ is 1 for each $j < d$.
6. Suppose F is a finite field with q elements and let $f(x) \in F[x]$ have degree d. Show that f is irreducible if and only if $f(x) \mid x^{q^d} - x$ and the monic gcd of $f(x)$ and $x^{q^j} - x$ is 1 for each $j < d$ with $j \mid d$.
7. For a finite field F, let $I(F, d)$ denote the number of monic irreducible polynomials in $F[x]$ of degree d. Compute $I(F, d)$ for each $F = \mathbf{Z}/(p)$ for $p = 2, 3, 5$ and for each $d = 1, 2, 3, 4, 5, 6$. For this problem you should use the identity (hopefully proved in class by Friday, April 11):

$$q^d = \sum_{j \mid d} jI(F, j) \quad \text{where} \quad \#F = q.$$

8. Suppose p is a prime and F is a finite field with p^{15} elements. We know that the polynomial $x^{p^{15}} - x$ has every element of F as a root. Let K be the set of roots in F of the polynomial $x^{p^3} - x$. Show that K is a subfield of F.