Proof of the Classification of Surfaces

Let us recall what we know about compact connected surfaces...

Theorem 1 The Enumerate Theorem Every compact connect surface is homeomorphic to the sphere (S^2), the connect sum of n tori ($\#^n T^2$), the projective plane (P^2), the Klein bottle (K), the connect sum of n tori and a projective plane $\#^n T^2 \# P^2$, or the connect sum of n tori and a Klein Bottle $\#^n T^2 \# K$.

In the problem below we call the first two types orientable the last four types non-orientable. Our goal in following exercises is to use the Van Kampen theorem to prove the **Classification Theorem**, namely that none of the surfaces appearing in the **Enumerate Theorem** are homeomorphic. This will involve computing the fundamental groups and developing enough algebra to tell that none of the fundamental groups which arise are isomorphic. In the process we will explore the first of the great tricks for understanding groups given in terms of generators and relations: the Abelianization trick.

Problem 2

1. Realize $\#^n T^2$ as an identification space using a $4n$-gon, realize $\#^n T^2 \# P^2$ as an identification space of using a $4n+2$-gon, and realize $\#^n T^2 \# K$ as an identification space of using a $4n+4$-gon.
2. Find the fundamental groups of all surfaces.
3. Describe a reason why you believe or do not believe that the gluing of this n-gon could be accomplished by a deck-like group of Euclidean isometries.

Now we'd like to understand some basic properties of these fundamental groups, which will entail understanding some basic "canonical subgroups" and there quotients. This is an expanded version of the original problem which is **not due** but that should be carefully thought about. Problem with a ($\star$) are unlikely to be explicitly used in the course but good practice.

Problem 3

1. **(Dealing With a Presentation)** Denote a presentation as $< X \mid R >$. Suppose that r is determined by R in the sense that it lies in $N(R)$. Let a type one Teitze move be a change of the presentation to $< X \mid R \bigcup \{r\} >$. Let a type two Teitze move be a change of the presentation to $< X \bigcup \{x\} \mid R \bigcup \{x^{-1}w\} >$ with w in a word in X.

(a) Prove that these moves preserve the group determined by these presentations up to isomorphism.

(b) ($\star$) Prove that given any two isomorphic finitely presented groups that one can express the first presentation in the second's form via a finite sequence of Teitze moves.

(c) ($\star$ and very hard, but you should think about what it might mean) There is no algorithm for deciding whether two finitely presented groups are isomorphic (there isn't evan an algorithm for deciding whether or not the group is trivial!).

2. Explore some examples.

   (a) Prove that $\times^n \mathbf{Z}$ is isomorphic to

$$< g_1, \ldots, g_n \mid \{g_i g_j g_i^{-1} g_j^{-1} \mid i, j \in \{1, \ldots, n\} > .$$

   (b) Prove that $\mathbf{Z}/n\mathbf{Z}$ is isomorphic to

$$< g \mid g^n > .$$

   (c) Prove $\times^n \mathbf{Z}$ is isomorphic to $\times^m \mathbf{Z}$ if and only if $n = m$.

   (d) ($\star$) Use the Teitze moves to show that $\mathbf{Z}/6\mathbf{Z}$ can be presented as

$$< a, b \mid aba^{-1}b^{-1}, a^2, b^3 > .$$

   (e) ($\star$) Show that S_3 (the group of permutations of three objects) is isomorphic to

$$< a, b \mid a^2, b^3, abab > .$$

3. Some canonical subgroups and quotients.

   (a) Let $[G, G] = N(\{aba^{-1}b^{-1} \mid a, b \in G\})$. Prove that $G/[G, G]$ is an Abelian group (called G's Abelianization),

   (b) If G is described in terms of generators and relations prove that its Abelianization is equivalent to adding in to its list of relations the relations $ghg^{-1}h^{-1}$ for all pairs of generators g and h.

   (c) Given an Abelian group A, let $Tor(A)$ be the set of all elements of finite order, i.e. the set of all $g \in A$ such that $g^n = id$ for some $n \in \mathbf{Z}\}$. Prove $Tor(A)$ is a normal subgroup of A.

   (d) ($\star$) If A is Abelian and finitely generated prove $A/Tor(A)$ is isomorphic to $\times^r \mathbf{Z}$, where r is called A's rank.

(e) Let

$$A = < g_1, \ldots, g_n \mid \{g_i g_j g_i^{-1} g_j^{-1} \mid i, j \in \{1, \ldots, n\} \bigcup \{g_{r+1}^{m_{r+1}}, \ldots, g_n^{m_n}\} >$$

and prove

$$Tor(A) = < g_{r+1}, \ldots, g_n \mid \{g_i g_j g_i^{-1} g_j^{-1} \mid i, j \in \{r, \ldots, n\} \bigcup \{g_{r+1}^{m_{r+1}}, \ldots, g_n^{m_n}\} >$$

and that the rank of A if r.

Problem 4 1. Find the Abelianization of the fundamental group for every surface, and find each Abelianization's torsion subgroup and rank.

2. Finish the **Classification theorem** by proving that none of the surfaces described in the **Enumeration Theorem** are homeomorphic.