Hyperbolic Geometry Problems

1. Prove that every element of of $\tilde{M}_{UHP}$ is an isometry of (UHP, ρ_{UHP}).
2. Let R be a region in (Ω, ρ) and let the area of R be defined to be
$$A(R) = \left| \int_R \rho^2(z) dx dy \right| .$$
Prove that this notion of area is an isometry invariant.
3. We call two geodesic tangent if they hit the same point at infinity. Prove that a set consisting of a horocirlce and two distinct geodesic which are all pairwise tangent is unique up to isometry.
4. Compute a finite length which arises naturally in the previous problem.
5. As in Euclidean geometry we will call a set in the hyperbolic plane convex if the geodesic connecting any two points in the set lies with in the set. Prove the intersection any collection of convex sets is convex.
6. Prove that a that a geodesic splits $H \bigcup \partial H$ into two closed convex regions.
7. The convex hull of a set S in $H \bigcup \partial H$ is the intersection of all half planes containing S (which is convex by the previous two problems). Prove that the convex hull of three points in $H \bigcup \partial H$ not lying on a geodesic forms a convex region bounded by precisely three geodesics (called a triangle).
8. Suppose we have a deck like group of isometries on H. Show for any point that the Dirichlet domain is a convex fundamental domain for the group action (see the initial handout for a review of the terminology).
9. (a) Let v be a vector at p, and prove there is a unique geodesic (up to parameterization) which is tangent to v at p.
   (b) For any $p \in H$ prove that an isometry f is determined by $f(p)$ and $f_\star(p)$.
   (c) Let $q, p \in H$, $\{v, w\}$ be orthonormal vectors at p, and $\{\hat{v}, \hat{w}\}$ be orthonormal vectors at q. Prove there exist a unique isometry f such that $f(p) = q$, $f_\star(v) = \hat{v}$ and $f_\star(w) = \hat{w}$.