1 Conclusion

1.1 The Geometric Deck Theorem Converse

Now we accomplish what we set out to do in the introduction. There we set out the following goals...

1. **Topological part of the solution:** In vast generality there is a converse to the Deck theorem. Namely we will find that "usually" a topological space can be expressed as the orbit space of a deck like action on a simply connected space (The converse to the Deck Theorem). This we have know accomplished together with an understanding of all the aspects of the Galois correspondence.

2. **Geometric part of the solution:**: We need to develop the geometric model spaces. We will only work in two dimension, where we are forced to come to grips with Euclidean, Spherical , and Hyperbolic geometry. In fact will let G be either the hyperbolic plane H^2, the Euclidean plane E^2, or the sphere S^2; and we have accomplished a good understanding of the most important and least familiar of these: hyperbolic space.

3. **Geometric topology part of the solution:** For compact surfaces we will attempt to realize the topological covering spaces as geometric model spaces with the deck group a subgroup of the isometry group. In the process we shall build geometric structures on all compact surfaces. We will also argue why the model geometry involved to build a given surface is unique. We know tackle this final goal.

Here I will list the result we proved in lecture. To start it off recall.

Definition 1 Let a G-surface surface M be a connected surface with a metric such that for each point $p \in M$ there is ball around it isometric to a ball in G (for a fixed G).

There is a special type of G-surface that will be of particular interest namely...

Definition 2 We say that G-surface is complete provided every curve that is locally a geodesic can be extended forever in both directions.

Now we acknowledge a fact out geometric structures and covers that will play fundamental role in connecting the geometry to the topology.

Lemma 3 Suppose ρ is a covering map from the connected surface M to the G-surface N, then M can be turned into a G-surface where ρ is a local isometry and where the deck transformations become isometries. Furthermore if N is complete then so is M.

(The proof of this lemma was discussed in lecture, and is can be viewed as a straightforward exercise to anyone who was not not present.)

Together with the Deck theorem we arrive at the following corollary of this lemma.

Corollary 4 Every complete G-surface is covered by a simply connected, complete G-surface where the deck transformations are isometries.

Now we like to get a grip on who these a simply connected, complete G-surfaces are.

Theorem 5 The Hopf-Rinow Theorem Every simply connected complete G-surface is isometric to S^2,H^2, or E^2.

(The proof was discussed in lecture and a reference to an arlternate equally basic proof is given on the course website.)

To apply the above results we need to identify some complete G-surfaces. One example is the following.

Lemma 6 Every compact G-surface is complete.

(The proof of this lemma was discussed in lecture, and is can be viewed as a straightforward exercise to anyone who was not not present.)

We can now sum it all up with...

Theorem 7 The Geometric Deck Theorem Converse Every compact G-surface is isometric to G/Γ where Γ is a group of isometries acting on G in a deck like way. Furthermore any construction of the surface universal cover will be isometric to G.

Sometimes this theorem is called the Hopf-Killing theorem.

1.2 The Compact G-surfaces

Now we will show that every compact surface can be given the structure of a G-surface. Notice that we have already done this for the torus and the Klein Bottle and that the anti-podal map is an isometry hence the sphere and projective plane are also immediately taken care of. The trick is the rest of them.

Recall the usual polygon used when forming a surface (like the first polygon shown in **picture 1**) and then rearrange it until we see the n-gon P (as in **picture 1**). It is this polygon that we will use now to construct the surface (and eventually it universal cover). To do so we will construct a hyperbolic polygon with such that each $(edge)^*$ has the same length as and $((edge)^\star)^{-1}$ and where the angles sum up to precisely 2π (as indicated in **picture 2**). By scaling a fixed regular n gon we see that we can in fact make a polygon with equal side length and achieve any angle sum between zero and $(n-2)\pi$, and in particular 2π when $n > 4$. Recalling from our construction of surfaces from n-gons that we used a $4n$-gon with $n \geq 2$ or $4n+2$-gon with $n \geq 1$ to form any surface M not equal to the torus, the Klein bottle, the sphere, or the projective plane we see that...

Theorem 8 Every compact connected surface can be made into a G-surface, and every surface M (as described above) can be made into an H^2-surface.

Now we'd like to explicitly see these surfaces. Now recall that we can form the universal cover of M from P via the quotient space of $\pi_1(M) \times P$ (where $\pi_1(M)$ is given the discrete topology) by the equivalence relation given by $(g, s) \equiv (ga_i, p_{a_i}^{-1}(s))$ when s is a member of $a_i^\star$, ect... (as indicated in **picture 1**). Using our hyperbolic version of P as above, we may use isometries when making all the above identifications hence turning this universal cover into a H^2-surface coving our H^2-surface M via a covering map which is a local isometry (call this a G-cover). Let us record this.

Lemma 9 Let M be as above, then $\pi_1(M) \times P/\equiv$ is the universal G-cover of M with the deck transformation given by $d[g, x] = [dg, x]$.

Now we look at the isometry of this universal cover to H^2, which we know exist by the Hopf-Rinow theorem. We start by isometrically embedding P. Now the deck transformations are isometries hence each is determined by where it sends a point and a half plane through the point. Notice this tells us explicitly how each element of $\pi_1(M)$ acts on H^2 and allows us to build the tessellation of H^2 via copies of P as in **picture 2** as well as how to explicitly produce the generators of our group of deck transformations as a subgroup of $Isom(H^2)$.