The Full Mobius Group and some nifty Sub-groups

The orientation preserving Mobius group M can be naturally extended via the following set of anti holomorphic mappings...

$$R = \left\{ \frac{a\bar{z}+b}{c\bar{z}+d} \mid a,b,c,d \in C; ad-bc \neq 0 \right\}.$$

Observe $\tilde{M} = M \bigcup R$ is also a group, called the Mobius group. (It is precisely the group of conformal homeomorphism of $\hat{C}$, while M is the group of orientation preserving conformal homeomorphisms). Notice that if f and g are in R then fg is in M. We have the following lemma concerning $\tilde{M}$....

Lemma 1 The Full Mobius Group lemma

1. $f \in R$ then f is an orientation reversing conformal homeomorphism of $\hat{C}$.
2. If $f \in \tilde{M}$ and $f \neq id$ then f has zero, one, two or a "circles" worth of fixed points.
3. For every "circle" there is a unique element of R fixing it point-wise.
4. If $f \in \tilde{M}$ then for every "circle" C, $f(C)$ is itself a "circle".

Exercise 2 If you know some group theory: Prove that $\tilde{M}$ is the semi direct product of $PGL(2,C)$ and $Z/2Z$ with the non-trivial element in $Z/2Z$ mapping to the conjugation automorphism of $PGL(2,C)$.

Now we are prepared to understand a pair of particularly nice subgroups of the Mobius group. Let $\tilde{M}_{UHP}$ be the sub-group of $\tilde{M}$ which preserves the upper-half plane, UHP; and let $\tilde{M}_{UD}$ be the sub-group of $\tilde{M}$ which preserves the unit disk, UD. When the tilde is remove we are asking for the corresponding subgroup of M instead of $\tilde{M}$. Furthermore let $S^{\star}L(2,R)$ be the elements of $GL(2,C)$ with real coefficients and determinant ± 1 and let $SL(2,R)$ be those $GL(2,C)$ elements with real coefficients and determinant $+1$. As before, let $PS^{\star}L(2,R) \cong S^{\star}L(2,R)/\{\pm I\}$ and $PSL(2,R) \cong SL(2,R)/\{\pm I\}$. Furthermore let Θ denote the mapping sending

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $S^{*}L(2,R)$ to $\frac{az+b}{cz+d}$ if A's determinant is $+1$ and sending A to $\frac{a\bar{z}+b}{c\bar{z}+d}$ if A's determinant is -1.

Exercise 3 Prove Θ is group homomorphism and that its image is isomorphic to $PS^{\star}L(2,R)$ when view as mapping from $S^{\star}L(2,R)$, and $PSL(2,R)$ when restricted to $SL(2,R)$.

We have the following lemma...

Lemma 4 The $\tilde{M}_{UHP}$ and $\tilde{M}_{UD}$ Identification Lemma

1. $\tilde{M}_{UHP}$ is isomorphic via Θ to $PS^{*}L(2,R)$.
2. The Cayley mapping,
$$\frac{-iz+i}{z+1},$$
is a conformal homeomorphism of UD onto UHP.
3. $\tilde{M}_{UHP} \cong \tilde{M}_{UD}$.
4. An element of M is in M_{UD} if only only if it can be expressed as
$$e^{i\theta}\frac{z-a}{\bar{a}z-1}.$$
with $\theta \in [0, 2\pi)$ and $a \in UD$.

Exercise 5 Describe a group of matrices which can be identified with the group M_{UD} via the correspondence described in exercise four of the conformal geometry handout (Hint use the Cayley transformation).

Since the groups above are isomorphic we will call the underlying groups M_H and $\tilde{M}_H$. Further more we shall let H denote either the UHP and UD, where statements involving H shall have two interpretations one in UHP and one in UD. When we discuss $g \in \tilde{M}_H$ let $\rho(g)$ be g's matrix representative as described in part 1 or 4 of the above lemma. Let ∂UD denote the unit circle, ∂UHP denote the real axis together with infinity, and let ∂H denote the one making sense of a given statement. For example here is a statement that can be interpreted in UD or UHP: call a mapping in M_H that fixes one point in H and no points in ∂H *elliptic*, a mapping which fixes no points in H and one in ∂H *parabolic*, and a mapping which fixes no points in H two points in ∂H *hyperbolic*. Here is a lemma presented in this language.

Lemma 6 The M_H Classification Lemma

1. Every mapping $g \neq id$ in M_H is either elliptic, parabolic, or hyperbolic.

2. The mapping $g \neq id$ in M_H is elliptic if $(tr(\rho(g)))^2 < 4$, parabolic if $(tr(\rho(g)))^2 = 4$, and hyperbolic if $(tr(\rho(g)))^2 > 4$.

3. The property of being elliptic, hyperbolic, or parabolic is invariant under conjugation; furthermore every element of of M_H is conjugate to either $z + b$ or $a^2 z$ (viewed in M_{UHP} with a and b real) or $e^{i\theta} z$ (viewed in M_{UD} with θ real).

Now let us look at the mappings in $\tilde{M}_H - M_H$. Call such a mapping a *glide reflection* if it fixes no points in H and two points of ∂H, and call such mapping a *reflection* if it fixes a "circle".

Exercise 7 1. Prove that every mapping in $\tilde{M}_H - M_H$ is a reflection of a glide reflection.

2. Prove that $g \in \tilde{M}_H - M_H$ is a reflection if $tr(\rho(g)) = 0$ and a glide reflection otherwise.

3. Prove that the property of being either a reflection or a glide reflection is invariant under conjugation, and that every element of of $\tilde{M}_H - M_H$ is conjugate to either the reflection $-\bar{z}$ or the glide reflection $-a^2\bar{z}$ (viewed in $\tilde{M}_{UHP}$ with a real).