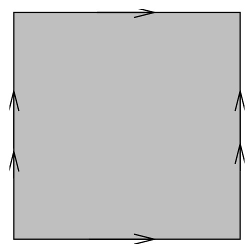

Figure 1: The torus.

1 Introduction and Review

1.1 Group Actions, Orbit Spaces and What Lies in Between

Our story begins with the torus, which we will think of initially as the identification space pictured in figure 1. Our first goal will be to recall that the torus can also be realized as the orbit space of a deck like group action on a simply connected space. Let us recall what a deck like group action is...

Definition 1 A Deck Like Action: Given a topological space Y and a group G given the discrete topology. Call a continuous group action of G on Y a deck like if for every $y \in Y$ there is an open set $U \subset Y$ such that $U \bigcap gU$ is empty for every $g \in G - id$.

(This is the language we used last quarter, but it is sadly not standard. One of our first tasks will be to understand some basic facts about group actions and develop the somewhat more complicated but standard language.)

In the case of the torus we let $Y = \mathbf{E}^2$ and the group be $G = \mathbf{Z} \times \mathbf{Z}$. The group action is given by $(m, n) \cdot (x, y) = (x + m, y + n)$. It is straight forward to verify that this particular group action is deck like, and you should convince your self of this before moving on. Furthermore you should recall why the orbit space $\mathbf{E}^2/\mathbf{Z} \times \mathbf{Z}$ is homeomorphic to the torus, see figure 2. The presence of a deck like action on a simply connected space allows us to explicitly understand the fundamental group of the torus. Namely $\pi_1(torus) \cong \mathbf{Z} \times \mathbf{Z}$. This follows form the Deck Theorem...

Theorem 2 (The Deck Theorem) If Y is simply connected and G is a deck like action on Y, then $\pi_1(Y/G)$ is isomorphic to G.

Recall that this isomorphism was straight forward to realize.

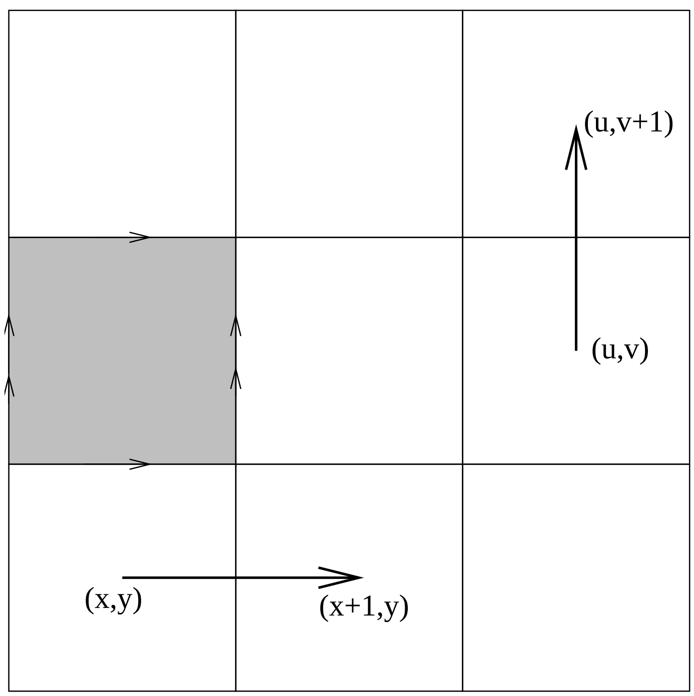

Figure 2: The orbit space. Here we see the action of a pair of generators $(1,0)$ and $(0,1)$ on $\mathbf{E}^2$.

The Isomorphism: Let the identification map be called $\pi : Y \to Y/G$. The isomorphism in the deck theorem is given by fixing $x \in Y/G$ and $y \in \pi^{-1}(x)$ then defining $\Psi(g) = < \pi(\lambda_g) >$ with λ_g any path connecting y and $g \cdot y$.

In figure 3 we see this notation in action.

In many way it is reasonable and worthwhile to view this construction of the torus upside down, i.e. suppose we started with the deck like group action provided by $\mathbf{Z} \times \mathbf{Z}$. We might well try and understand this group by looking at the orbit space, but it often proves convenient to look at an idea lives between the group action and its orbit space, called the group's fundamental domain.

Definition 3 The fundamental region of a group action on X is a closed set F which is the closure its interior F^o and satisfies

1. $\bigcup_{g \in G}(g \cdot F) = X$
2. $F^o \bigcap (g \cdot F^o)$ is empty for every non-identity $g \in G$.

Notice that a fundamental region of the group action described above and pictured in figure 2 is the grey square. We will see that the orbit space of space of a deck like action can be obtained by gluing up the boundary of the fundamental region using the group action. For the torus this corresponds to the identification space in figure 1. The pattern left over after translation moving a fundamental region, the pattern of squares for our torus, has a name as well.

Definition 4 If F is a fundamental region the $\{g \cdot F \mid g \in G\}$ is called a tessellation.

1.2 Covering Spaces

Now we will rephrase this discussion in the language of covering spaces. First let us recall what a covering space is (this definition will be from Hatcher, which differs slightly form Armstrong).

Definition 5 We say $\tilde{X}$ covers X if there is a continuous map $p : \tilde{X} \to X$ and an open cover of X such that for every U in the cover $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$ each of which is mapped by p homeomorphically to U. p is a called the covering map.

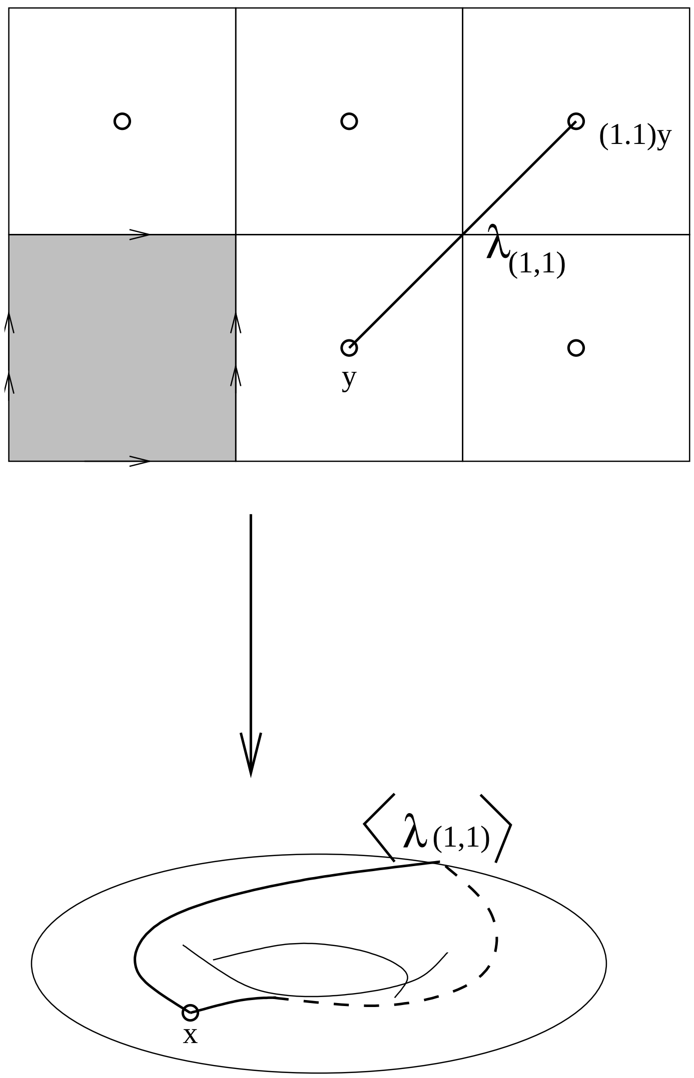

Figure 3: Fix y in $\pi^{-1}(x)$ and we notice that the fundamental group element corresponding to $g = (1,1)$ is $\pi(\lambda_{(0,1)}$ as pictured.

From the above construction we may observe that the plane covers the torus and we may view the identification map as the covering map. This follows from the following lemma from basic topology.

Lemma 6 If G acts on Y in a deck like way then then Y covers Y/G and with covering map is given by the identification map.

Continuing in this vain let us rephrase the Deck theorem and deck like actions in terms of covering space. This will involve some legitimately new language.

Definition 7 Let $\tilde{X}$ be a covering of X with covering map denoted by p. A homeomorphism f of $\tilde{X}$ is called a deck transformation if $p(f(x)) = p(x)$ for every $x \in \tilde{X}$. The set of all deck transformations forms a group called the Deck group and denoted $G(\tilde{X})$

Problem 8

1. Prove that the Deck group is indeed a group.
2. Explain why the deck group associated to the covering of the the torus by the plane is precisely $\mathbf{Z} \times \mathbf{Z}$ (prove it if you can).

In the language of Deck transformations we will find that the translation of the Deck theorem becomes...

Theorem 9 (Rephrasing the Deck Theorem) If $\tilde{X}$ is a simply connected cover of X then, then $\pi_1(X)$ is isomorphic to $G(\tilde{X})$.

The main connection to the work in the last section is that nearly every topological space fits into this context!

Theorem 10 (Converse of the Deck Theorem) If X is a nice (path connected, locally path connected, and semi-locally simply connected) topological space then there exist a simply connected space $\tilde{X}$ covering X. Furthermore $\pi_1(X)$ is isomorphic to $G(\tilde{X})$ and X is homeomorphic to the orbit space $\tilde{X}/G(\tilde{X})$.

1.3 Where the Geometry Fits In

Now we will take a closer look at our example. $\mathbf{E}^2$ has more structure than merely a topological space, it of course also has the structure of a "geometry". This notion of "geometry" contains many underlying ideas. For example we may wish to capture the notions of distance, angle, straight line, the length of a curve, isometries (congruences) ect.... In the case of $\mathbf{E}^2$ all these notions are familiar and there are many ways to develop them. $\mathbf{E}^2$ will serve as our first example of a "geometric model space". We will develop these geometric notions more carefully later, but for now you can think of a "geometric model space" as a topological space where the above geometric notions make and there are lots and lots of isometries. Recall that isometries are homeomorphism which also preserve all these geometric concepts. In the case of $\mathbf{E}^2$ we called this group $E(2)$ and it served as one of our main examples of a topological group in basic topology. Notice that $G = \mathbf{Z} \times \mathbf{Z}$ is a subgroup of $E(2)$.

One advantage of playing with group action in the presence of geometry is there becomes various canonical ways to find fundamental regions and tessellation. For example...

Definition 11 Suppose we have a group acting by isometries on a "geometric model space" X. Let

$$D_p(G) = \{z \in X \mid d(z,p) \leq d(z, g \cdot p) \mathtt{forallg} \in \mathtt{G}\}.$$

This is called the Dirichlet region with respect to the point p.

For deck like action as well as many other types of actions this the Dirichlet region is a fundamental region (in fact a very nice one).

Problem 12 Show that in the case of our $\mathbf{Z} \times \mathbf{Z}$ action on $\mathbf{E}^2$ that the D_p is a convex fundamental domain with respect to any point and describe it.

Since our action on $\mathbf{E}^2$ used to form the torus is via isometries will say that the torus has been given an "Euclidean structure". In the next problem we will explore why this is fair to assert.

Problem 13

1. Let x and y be two points on the torus and let the distance between x and y, $d(x,y)$, be defined by the greatest lower bound of the set of values $\{|\tilde{x} - \tilde{y}|\}$ with $\tilde{x}$ and $\tilde{y}$ lifts of x and y. Show that $d(x,y)$ a metric.
2. Show that every point of the torus has a ball in the metric constructed above which is isometric to a ball in $\mathbf{E}^2$.

In general we define the following.

Definition 14 A "geometric structure" on a topological space X is a metric on X whose topology agrees with X's topology and satisfies that each point of X has a ball around it isometric to a ball in some fixed "geometric model space".

By using the converse to the Deck theorem we will see that there is very often a second way to view such a structure. This view point intimately intertwines the fundamental concepts being explored in this class namely discrete groups, covering spaces, and geometry,

fact 15 A "geometric structure" on a topological space is "usually" equivalent to covering the space by a "geometric model space" where the deck transformations are isometries.

In the presence of such a cover the argument for the existence of a "geometric structure" will analogous to the argument used in the problem above. The converse will require the Deck Theorem's converse.

1.4 Where Do We Go From Here

Summary: The topological space the torus has a "geometric structure", namely it can be covered by $\mathbf{E}^2$ in such a way that the group of deck transformation is a subgroup of $E(2)$.

The Geometrization Problem: Given a topological space find a "geometric structure" on the space, and decide whether the space can admit multiple "geometric model spaces".

Nature of solution that we will explore:

1. **Topological part of the solution:** In vast generality there is a converse to the Deck theorem. Namely we will find that "usually" a topological can be expressed as the orbit space of a deck like action on a simply connected space (The converse to the Deck Theorem). This will involve developing some covering space theory.

2. **Geometric part of the solution:**: We need to develop the geometric model spaces. We will only work in two dimension, where we are forced to come to grips with Euclidean, Spherical , and Hyperbolic geometry.

3. **Geometric topology part of the solution:** For compact surfaces we will attempt to realize the topological covering spaces as geometric model spaces with the deck group a subgroup of the isometry group. In the process we shall build geometric structures on all compact surfaces.

We will also argue why the model geometry involved to build a given surface is unique.