Mathematics 74-114 Final Examination
Spring 2012

This examination is a take-home test. It consists of three problems of varying difficulty and length. You are to do your own work and not discuss the exam with anyone. However, inform the instuctor if you believe that you have found a typo or mistake. For sources you may use the textbook, your homework or your class notes. You may cite a result without proof if it appears in either (1) the assigned reading in the text (2) the assigned homework (3) your class notes. No other sources are permitted. Even if you cannot solve one part of a problem, you may still use that result in a later part of the problem.
Important: Write on one side of the paper and show all work. Give a reason for your assertions (except the obvious and well-known ones), but try to keep your solutions short. To receive full credit you must explain your work clearly and write it legibly.

I. In this problem let $\simeq$ denote chain homotopy of chain maps. Let $C = \{C_n, \partial_n\}$ be a chain complex, $n \geq 0$. Define $\widehat{C} = \{\widehat{C}_n, \widehat{\partial}_n\}$ and $C^+ = \{C^+_n, \partial^+_n\}$ by

$$\widehat{C}_n = C_n \oplus C_{n-1} \quad \text{and} \quad \widehat{\partial}(x,y) = (\partial x + y, -\partial y)$$

$$C^+_n = C_{n-1} \quad \text{and} \quad \partial^+_n = -\partial_{n-1}.$$

Clearly C^+ is a chain complex. Show

(a) $\widehat{C}$ is a chain complex.

(b) Use the chain complexes C, C^+ and $\widehat{C}$ to show that for all $q \geq 0$,

$$H_q(\widehat{C}) = 0.$$

(c) Prove that $\mathrm{id} : \widehat{C} \to \widehat{C}$ is chain homotopic to the zero homomorphism $0 : \widehat{C} \to \widehat{C}$ if C_n is free-abelian for all n. In your proof you can use the Lemma 2.9 (p. 260), but if you do, you are to write out its proof and supply all missing details

(d) Prove the following: If $f : C \to D$ is a chain map and $f \simeq 0$, then there exists a chain map $\widehat{f} : \widehat{C} \to D$ extending f (i.e., $\widehat{f}(x, 0) = f(x)$.)

II. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "p_n"] \arrow[d, "g_n"] & C_n \arrow[r, "d_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "p'_n"] & C'_n \arrow[r, "d'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where every third homomorphism h_n is an isomorphism.

(a) Prove that the following sequence is exact

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{d_n h_n^{-1} p'_n} A_{n-1} \longrightarrow \cdots$$

where $(i_n, f_n)(a) = (i_n a, f_n a)$ and $(g_n - i'_n)(b, a') = g_n b - i'_n a'$.

(b) Let X_1 and X_2 be subspaces of a space X such that $X = X_1^o \cup X_2^o$. Use part (a) to prove that the following sequence is exact

$$\cdots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(j_{1*}, j_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_* - k_*}$$

$$H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

where j_1, j_2, j and k are inclusions and $D = \Delta h_*^{-1} l_*$, where the maps $l : (X, \emptyset) \to (X, X_2)$ and $h : (X_1, X_1 \cap X_2) \to (X, X_2)$ are inclusions of pairs and Δ is the connecting homomorphism of the homology of $(X_1, X_1 \cap X_2)$. Do not use the $\mathcal{V}$-small theorem. This gives another proof of the Mayer-Vietoris sequence (see Homework 8, problem 12). A slight modification of the proof gives the above exact sequence for reduced groups. (You need not prove this.)

(c) Define the suspension of a space X by $SX = X \times I/\sim$ where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$, for all $x, x' \in X$. Prove that $H_{q+1}(SX) \approx \widetilde{H}_q(X)$, $q \geq 0$. (Hint: Find two contractible subspaces.)

III. Let K be a CW-complex.

(a) Prove that $H_l(K) \approx H_l(K^r)$ if $l + 1 \leq r$.

(b) Prove that $H_l(K)$ is free-abelian if l is the dimension of K.

(c) Show that K is path-connected if and only if K^1 is path-connected.

(d) Massey, p. 206, problem 4.2. Construct the space X as a CW-complex and prove that $H_1(X) \approx \mathbb{Z}_m$. (Hint: See problem 4.1.)