Handout #4. Isometries of $\mathbb{R}^n$

4. Isometries of $\mathbb{R}^n$

Exercise 4.1. Show that an orthogonal set is linearly independent.

Exercise 4.2. Let O_n be the set of all *orthogonal matrices*:

$$O_n = \left\{ A \in M_n(\mathbb{R}) \;\middle|\; A^t \cdot A = A \cdot A^t = I_n \right\}.$$

Show that O_n is a group with respect with matrix multiplication.

Definition. A map $h : \mathbb{R}^n \to \mathbb{R}^n$ is called an *Euclidean isometry* if

$$\| h(\mathbf{x}) - h(\mathbf{y}) \| = \| \mathbf{x} - \mathbf{y} \|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Exercise 4.3. Let $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(\mathbf{0}) = \mathbf{0}$.

(a) h is an isometry if and only if it preserves dot products, that is

$$\langle h(\mathbf{x}), h(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

(b) h is an isometry if and only if it is an orthogonal transformation.

Exercise 4.4. In general, $h : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if

$$h(\mathbf{x}) = A \cdot \mathbf{x} + \mathbf{p}, \text{ where } A \in O_n \text{ and } \mathbf{p} \in \mathbb{R}^n.$$

Exercise 4.5. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. If $S \in \mathbb{R}^n$ is rectifiable, then $T = h(S)$ is rectifiable, and $v(T) = v(S)$.