Math 73, Winter 2004, Final Exam

due Tuesday, March 16, noon

Instructor: Dorin Dumitraşcu

Instructions: You may use the text-book, your notes, or the library, but the written form that you hand in should be well articulated and coherent, and should reflect your understanding of the assigned problems. **The only** person you may discuss the exam with is your instructor. A violation of this will be treated as a violation of the Honor Principle.

There are two sections. For perfect score you should do five problems from each section. The graduate students must do the last two problems from each section.

Write on one side of your paper, each problem on a separate sheet, and make sure that your name is on every page. Include this page with your solutions. Bring the exam to my office or slide it under my door before the due date.

1.1. ______ /10	1.2. ______ /10	1.3. ______ /10
1.4. ______ /10	1.5. ______ /10	1.6. ______ /10
2.1. ______ /10	2.2. ______ /10	2.3. ______ /10
2.4. ______ /10	2.5. ______ /10	2.6. ______ /10

Total: ______ /100

1. Manifolds

Exercise 1.1. 2, p.70.

Exercise 1.2. 1, p.193.

Exercise 1.3. Let λ_n be the volume of the unit ball in $\mathbb{R}^n$. (There is a formula for this but you *do not* need to recall or prove it here.) Use the Change of Variables Theorem to express the volume of the ellipsoid

$$\frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1,$$

where $a_1, \ldots, a_n$ are positive real constants, as the integral of an appropriate function over the unit ball in $\mathbb{R}^n$. Then evaluate the integral. (The answer will involve λ_n.)

Exercise 1.4.
(a) Prove that the hyperboloid in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = a^2$ is a manifold. (Here $a > 0$ is a constant.) Find the tangent space to this manifold at the point $(a, 0, 0)$.
(b) Does the equation $x^2 + y^2 - z^2 = 0$ define a manifold? Explain briefly.

Exercise 1.5. Let $M \subset \mathbb{R}^n$ be a smooth compact k-manifold, $k > 1$. Let $r : \mathbb{R}^k \to \mathbb{R}^k$ be the mapping that sends $(x_1, x_2 \ldots, x_k)$ to $(-x_1, x_2, \ldots, x_k)$. Let $\alpha_0 : U_0 \to V_0 \subset M$ and $\alpha_1 : U_1 \to V_1 \subset M$ be two coordinate patches that overlap negatively. Show that α_0 and $\alpha_1 \circ r : r(U_1) \to V_1 \subset M$ overlap positively. Use this to show that every manifold that is coverable by two connected coordinate patches is orientable.

Exercise 1.6. 8, p.292.

2. Forms; Integration of forms

Exercise 2.1. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x^2 e^y, x - y)$, and $\omega = x\, dy \in \Omega^1(\mathbb{R}^2)$.
(a) Evaluate $\omega(\mathbf{p})(\mathbf{p}; \mathbf{v})$, where $\mathbf{p} = (1, 0)$ and $\mathbf{v} = (3, 5) = 3\mathbf{e}_1 + 5\mathbf{e}_2$.
(b) Compute $f^*\omega$ and $d\, f^*\omega$.
(c) Verify that $d\, f^*\omega = f^*\, d\omega$.

Exercise 2.2. Let $U = \{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1 \}$ be the unit disk in $\mathbb{R}^2$. Denote by $\mathbf{x} = (x_1, x_2)$ the generic point in $\mathbb{R}^2$, and by $\mathbf{y} = (y_1, y_2, y_3, y_4)$ the generic point in $\mathbb{R}^4$. Let $\alpha : U \to \mathbb{R}^4$ be defined by $\alpha(\mathbf{x}) = (0, x_1, x_2, 0)$.

(a) Compute for each $\omega_{ij} = dy_i \wedge dy_j$, $1 \le i < j \le 4$, the 2-from $\alpha^*\omega$.

(b) Let $M = \alpha(U)$. Compute

$$I_{12} = \int_M \omega_{12} \qquad \text{and} \qquad I_{23} = \int_M \omega_{23}.$$

Exercise 2.3. Let M be a 2-dimensional compact oriented C^∞-manifold in $\mathbb{R}^4$ without boundary. Consider the 2-forms

$$\omega_1 = (x_2 \sin(x_1 x_2) + x_3)\, dx_1 \wedge dx_3$$

and

$$\omega_2 = -x_1 \sin(x_1 x_2)\, dx_2 \wedge dx_3 + x_3\, dx_1 \wedge dx_3.$$

Which integral is bigger: $\int_M \omega_1$ or $\int_M \omega_2$? Explain your answer.

Hint. Use the algebraic properties of the differential operator d and of the integral over manifolds. No complicated computations are needed; Stokes' theorem may be useful.

Exercise 2.4. Let $\mathbb{R}^3$ have coordinates (x, y, t). Suppose $u = u(x, y, t)$ is a C^∞ solution of the **heat equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

in an open set A of $\mathbb{R}^3$. Define the differential 2-form α on A by

$$\alpha = \left(\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right) \wedge dt - u\, dx \wedge dy$$

Show that if $M \subset A$ is any compact oriented 3-manifold with boundary Σ, then

$$\int_\Sigma \alpha = 0.$$

Exercise 2.5. Let M be a k-dimensional oriented manifold in $\mathbb{R}^n$, with or without boundary. Let ω be a k-form on M, with $C = \operatorname{supp}(\omega)$ compact. Assume that there are two positively oriented charts $\alpha : U_\alpha \to V_\alpha$ and $\beta : U_\beta \to V_\beta$ such that $C = \operatorname{supp}(\omega) \subset V_\alpha \bigcap V_\beta$. Show that

$$\int_{U_\alpha} \alpha^* \omega = \int_{U_\beta} \beta^* \omega.$$

Conclude that the integral $\displaystyle\int_M \omega$ is well-defined.

Exercise 2.6. Prove Theorem 35.2.