Interesting problems, II

You may turn in any of these sets for extra credit. The due date is March 12, or before. You may work individually or in group. Only turn these in if you obtain results.

2. Differential forms

Exercise 2.1. (a) Do Exercise 5, page 261 in the text-book.

(b) Show that on $\mathbb{R}^n$ every closed form is exact. You may want to wait till we prove Stokes' theorem for this part. You will have to relate somehow the initial definitions with integration.

Exercise 2.2. Prove the following:

Theorem. *A k-dimensional compact manifold without boundary in $\mathbb{R}^n$ is orientable if and only if there exists a k-form ω on M such that $\omega(\mathbf{p}) \neq 0$ for all $\mathbf{p} \in M$.*

Below are some hints on how to proceed.

(i) Suppose first that such ω exists. Show that, for any coordinate patch $\alpha : U \to V$ on M, $\alpha^*\omega$ is a nowhere vanishing k-form on the open set $U \subset \mathbb{R}^k$. Consequently there exists a nowhere zero C^∞ function $f : U \to \mathbb{R}$ such that $\alpha^*\omega = f\, dx_1 \wedge \cdots \wedge dx_k$. (Why?) Define the coordinate patch to be *positively oriented* if $f > 0$ and *negatively oriented* if $f < 0$.

(ii) If $\alpha : U \to V$ is negatively oriented, show how to modify U and α to obtain a positively oriented patch $\alpha' : U' \to V$ (same image V). Thus M can be covered by positively oriented patches.

(iii) Show that two overlapping positively oriented patches overlap positively (as we discussed in class). Thus M is orientable.

(iv) For the converse, assume that M is orientable and cover M with coordinate patches $\alpha_i : U_i \to V_i$ such that all pairs of overlapping patches overlap positively. On each V_i define a k-form ω_i by

$$\omega_i = (\alpha_i^{-1})^*(dx_1 \wedge \cdots \wedge dx_k).$$

Each V_i is the intersection of M with an open set $W_i \subset \mathbb{R}^n$. Let W_0 be $\mathbb{R}^n \setminus M$, such that $\{W_0, W_1, W_2, \ldots\}$ covers $\mathbb{R}^n$. By Theorem 16.3, there is a partition of unity $\{\phi_i\}$ on $\mathbb{R}^n$ such that $\operatorname{supp}(\phi_i) \subset W_i$, for $i = 0, 1, 2, \ldots$. Extend $\phi_i\omega_i$ to a k-form σ_i on M, by defining σ_i to be 0 off V_i (and of course $\sigma_i = \phi_i\omega_i$ on V_i). Let $\omega = \sum_{i=1}^{\infty} \sigma_i$. Verify that ω is a nowhere vanishing k-form.