Interesting problems, I

You may turn in any of these sets for extra credit. The due date is March 12, or before. You may work individually or in group. Only turn these in if you obtain results.

1. The tangent space to $\mathbb{R}^n$ at a point

The following discussion is meant to justify the notation $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dots, \dfrac{\partial}{\partial x_n}$, used to designate the basis of the tangent space $T_{\mathbf{x}}\mathbb{R}^n$.

Let $C^\infty(\mathbb{R}^n)$ be the space of all C^∞ real-valued functions on $\mathbb{R}^n$. Let $\mathbf{p}$ be a point in $\mathbb{R}^n$. A **derivation at p** is a linear map $D : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = D(f)\, g(\mathbf{p}) + f(\mathbf{p})\, D(g).$$

Exercise 1.1. Show that every tangent vector at $\mathbf{p}$,

$$(\mathbf{p}, \mathbf{v}) = (\mathbf{p}, a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n}),$$

defines a derivation at $\mathbf{p}$ by the formula:

$$D(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(\mathbf{p}) = f'(\mathbf{p}; \mathbf{v}).$$

(*Note.* You may want to start with $\mathbf{v} = \mathbf{e}_i = \dfrac{\partial}{\partial x_i}$ and extend afterwards by linearity.)

Amazingly enough the converse also holds. Try to prove:

Theorem. *Every derivation at* $\mathbf{p}$ *is a directional derivative, that is, given a derivation* D*, there exists* $\mathbf{v} \in \mathbb{R}^n$ *such that* $D(f) = f'(\mathbf{p}; \mathbf{v})$.

Here are two hints:

Exercise 1.2. Show that if $f \in C^\infty(\mathbb{R}^n)$ and $f(\mathbf{0}) = 0$, then there exist functions $g_i \in C^\infty(\mathbb{R}^n)$ such that

$$g_i(\mathbf{0}) = D_i(f)(\mathbf{0}) \quad \text{and } f(\mathbf{x}) = \sum_{i=1}^{n} x_i\, g_i(\mathbf{x}).$$

(*Hint.* Write $f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dt$.)

Exercise 1.3. Show that if D is a derivation, then $D(f) = 0$ whenever f is a constant function.

To prove the Theorem, start by assuming that $\mathbf{p} = \mathbf{0}$. Then use 6.3 to reduce to the case $f(\mathbf{0}) = 0$, and use 6.2 to complete the argument in this particular case. Finally do the wrap up.