Math 73/103 Midterm

1. (15) Give precise statements (no proofs necessary on this problem) of Littlewood's Three Principles:

(a) Every Lebesgue measurable set is almost a disjoint union of intervals.

(b) Every sequence of Lebesgue measurable functions that converges almost everywhere is nearly uniformly convergent.

(c) Every Lebesgue measurable function is nearly continuous.

2. Let $(\mathbf{R}, \mathfrak{M}, m)$ be Lebesgue measure. Recall that $E \in \mathfrak{M}$ if and only if $E + y \in \mathfrak{M}$ for all $y \in \mathbf{R}$, and that $m(E) = m(E + y)$.

(a) Let $f \in \mathcal{L}^1(m)$ and $y \in \mathbf{R}$. Define $g(x) = f(x - y)$. Show that $g \in \mathcal{L}^1(m)$ and that

$$\int_{\mathbf{R}} f(x)\,dm(x) = \int_{\mathbf{R}} f(x - y)\,dm(x).$$

(b) If $f \in \mathcal{L}^1(m)$, let $\lambda_y(f) \in \mathcal{L}^1(m)$ be given by $\lambda_y(f)(x) = f(x - y)$. Show that $y \mapsto \lambda_y(f)$ is continuous from $\mathbf{R}$ to $L^1(m)$ in the sense that if $y_n \to y$ in $\mathbf{R}$, then $\|\lambda_{y_n}(f) - \lambda_y(f)\|_1 \to 0$.

(Hint: in part (a) start with characteristic functions. In part (b), you can reduce to the case where $y = 0$, and the conclusion is not so hard if f is continuous and vanishes off a bounded interval.)

3. Recall that if X is a topological space, then $\mathfrak{B}(X)$ is the σ-algebra of Borel sets in X. Show that $\mathfrak{B}(\mathbf{R}^2) = \mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$.

4. Suppose that f and g are functions from $\mathbf{R}$ to $\mathbf{R}$ with f Lebesgue measurable and g Borel. Which of $g \circ f$ and $f \circ g$ must be Lebesgue measurable? Why? (You need not deal with the other case.)

5. Suppose that $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$ and that f is also continuous. Is it necessarily true that $\lim_{x\to\infty} f(x) = 0$?

6. Carefully state the Monotone Convergence Theorem and Fatou's Lemma for non-negative functions. What happens if drop the hypothesis that each $f_n \geq 0$? Justify your assertions.

7. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable.

(a) Show that $F : (\mathbf{R}^2, \mathfrak{M} \otimes \mathfrak{M}) \to (\mathbf{R}^2, \mathfrak{B}(\mathbf{R}^2))$ given by $F(x, y) = \big(f(x), y\big)$ is measurable. (This just means that $F^{-1}(V) \in \mathfrak{M} \otimes \mathfrak{M}$ when V is open in $\mathbf{R}^2$.)

(b) Show that

$$G(f) = \{\, \big(x, f(x)\big) \in \mathbf{R}^2 : x \in \mathbf{R}\,\}$$

is in $\mathfrak{M} \otimes \mathfrak{M}$.

(c) Show that for almost all y,

$$m\big(\{\, x \in \mathbf{R} : f(x) = y\,\}\big) = 0.$$

(Hint: all these parts are connected. Any if you were to use something like Tonelli or Fubini's Theorem, you should carefully explain how.)