Math 73/103 Assignment Three
Due Friday, November 4th

Clarification: Since at least one person found some legitimate ambiguities in their notes, let me be clear about our terminology. Lebesgue measure, $(\mathbf{R}, \mathfrak{M}, m)$, is the complete measure coming from the explicit outer measure m^* we defined in lecture. In particular, $\mathfrak{M}$ is the σ-algebra of all m^*-measurable sets. A Lebesgue measurable function $f : \mathbf{R} \to \mathbf{C}$ is just a function such that $f^{-1}(V) \in \mathfrak{M}$ for any open set $V \subset \mathbf{C}$. We say f is Borel if $f^{-1}(V)$ is a Borel set in $\mathbf{R}$ for every open set V. We say $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$, or the f is Lebesgue integrable, if f is measurable and $\int_{\mathbf{R}} |f|\, dm < \infty$. We have also used the notation $L^+(\mathbf{R}, \mathfrak{M}, m)$ for the collection Lebesgue measurable functions f such that $f \geq 0$ everywhere.

1. Suppose that $f \in \mathcal{L}^1(X, \mathfrak{M}, m)$ is a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Show that there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \epsilon$.

2. Prove Lusin's Theorem: Suppose that f is a Lebesgue measurable function on $[a, b] \subset \mathbf{R}$. Given $\epsilon > 0$, show that there is a closed subset $K \subset [a, b]$ such that $m([a, b] \setminus K) < \epsilon$ and that $f|_K$ is continuous. (And unlike the version stated in lecture, we are not assuming f is integrable.)

3. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X. Let ρ^* be the associated outer measure.

(a) Show that $\rho^*(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathfrak{M}^*$ is the σ-algebra of ρ^*-measurable sets, show that $\mathcal{A} \subset \mathfrak{M}^*$.

4. Suppose that $f_n \to f$ in measure and that there is a $g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ is such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Show that $f_n \to f$ in $L^1(X, \mathfrak{M}, \mu)$.

5. Let m be Lebesgue measure on $[0, 1]$ and let μ be counting measure. Clearly, $m \ll \mu$. Show that there is no function f satisfying the conclusion of the Radon-Nikodym Theorem. Why is this not a counter-example to the Radon-Nikodym Theorem.

6. Prove the version of Fubini and Tonelli for complete measures stated in lecture: Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be *complete* σ-finite measure spaces. Let $(X \times Y, \mathfrak{L}, \lambda)$ be the completion of $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$. Suppose that f is $\mathfrak{L}$-measurable and that either (a) $f \geq 0$ or (b) $f \in \mathcal{L}^1(\lambda)$. Show that f_x and f^y are measurable almost everywhere and in case (b), then they are integrable almost everywhere. And, with suitable modifications on null sets, $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are measurable and even integrable in case (b). Then show that the iterated integrals both agree with the double integral.

(Here is what I suggest, let g be a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function that equals f almost everywhere. Then prove the following lemmas:

(a) If $E \in \mathfrak{M} \otimes \mathfrak{N}$, and $\mu \times \nu(E) = 0$, then $\nu(E_x) = 0 = \mu(E^y)$ for almost all x and y.

(b) If f is $\mathfrak{L}$-measurable and $f = 0$ λ-almost everywhere, then f_x and f^y are integrable almost everywhere and $\int_X f^y \, d\mu = 0 = \int_Y f_x \, d\nu$.)

7. Let ν be a complex measure on $(X, \mathfrak{M})$.

(a) Show that there is a measure μ and a measurable function $\varphi : X \to \mathbf{C}$ so that $|\varphi| = 1$, and such that for all $E \in \mathfrak{M}$,

$$\nu(E) = \int_E \varphi \, d\mu. \tag{$\dagger$}$$

(Hint: write $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ for measures ν_i. Put $\mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4$. Then μ_0 will satisfy ($\dagger$) provided we don't require $|\varphi| = 1$. You can then use without proof the fact that any complex-valued measurable function h can be written as $h = \varphi \cdot |h|$ with φ unimodular and measurable.)

(b) [Optional: Do not turn in] Show that the measure μ above is unique, and that φ is determined almost everywhere $[\mu]$. (Hint: if μ' and φ' also satisfy ($\dagger$), then show that $\mu' \ll \mu$, and that $\frac{d\mu'}{d\mu} = 1$ a.e. Also note that if φ' is unimodular and $E \in \mathfrak{M}$, then $E = \bigcup_{i=1}^4 E_i$ where $E_1 = \{ x \in E : \operatorname{Re} \varphi' > 0 \}$, $E_2 = \{ x \in E : \operatorname{Re} \varphi' < 0 \}$, $E_3 = \{ x \in E : \operatorname{Im} \varphi' > 0 \}$, and $E_4 = \{ x \in E : \operatorname{Im} \varphi' < 0 \}$.)

Comment: the measure μ in question 7 is called the *total variation* of ν, and the usual notation is $|\nu|$. It is defined by different methods in your text: see chapter 6. One can prove facts like $|\nu|(E) \geq |\nu(E)|$, although one doesn't always have $|\nu|(E) = |\nu(E)|$; this also proves that even classical notation can be unfortunate.

8. [Optional: Do NOT turn in] Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function. We want to show that f is Riemann integrable if and only if $m\big(\{\, x \in [a, b] : f \text{ is not continuous at } x \,\}\big) = 0$. In [1, Theorem 2.28], Folland suggests the following strategy. Let

$$H(x) = \lim_{\delta \to 0}\big(\sup\{\, f(y) : |y - x| \le \delta \,\}\big) \quad \text{and} \quad h(x) = \lim_{\delta \to 0} \inf\{\, f(y) : |y - x| \le \delta \,\}.$$

(a) Show that f is continuous at x if and only if $H(x) = h(x)$.

(b) In the notation of our proof in lecture that Riemann integral functions are Lebesgue integrable, show that $h = \ell$ almost everywhere and $H = u$ almost everywhere.

(c) Conclude that $\int_a^b h \, dm = \mathcal{R}\underline{\int}_a^b f$ and $\int_a^b H \, dm = \mathcal{R}\overline{\int}_a^b f$.

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. MR2000c:00001