Math 73/103: Homework on the Cauchy-Riemann Equations
Due TBA

1. Suppose that Ω is a *region* in $\mathbf{C}$, and that $f \in H(\Omega)$. Show that if $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

Let Ω be a domain in $\mathbf{C}$ and assume that $f : \Omega \to \mathbf{C}$ is a function. Of course, we can view Ω as an open subset of $\mathbf{R}^2$ and define $u, v : \Omega \to \mathbf{R}$ by

$$u(x, y) := \operatorname{Re}\bigl(f(x + iy)\bigr) \quad \text{and} \quad v(x, y) = \operatorname{Im}\bigl(f(x + iy)\bigr)$$

We say that the *Cauchy-Riemann Equations hold at* $z_0 = x_0 + iy_0$ if the partial derivatives of u and v exist at (x_0, y_0) and

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{CR}$$

We often abuse notation slightly, and say that (CR) amounts to $f_y(z_0) = if_x(z_0)$. (Just to be specific, $f_x(x_0 + iy_0) := u_x(x_0, y_0) + iv_x(x_0, y_0)$.)

2. Suppose that $f'(z_0)$ exists. Show that

$$f_x(z_0) = f'(z_0) = -if_y(z_0). \tag{1}$$

Conclude that the Cauchy-Riemann equations hold at z_0 whenever $f'(z_0)$ exists. Verify (1) when $f(z) = z^2$.

3. Suppose that Ω is a region and $f \in H(\Omega)$. Show that if f is real-valued in Ω, then f is constant.

4. Suppose that Ω is a region and $f \in H(\Omega)$. Suppose that $z \mapsto |f(z)|$ is constant on Ω. Show that f must be constant. (Consider $|f(z)|^2$.)

We let f, u, v and Ω be as above. Define

$$F:\Omega\subset\mathbf{R}^2\to\mathbf{R}^2 \quad\text{by}\quad F(x,y)=\big(u(x,y),v(x,y)\big).$$

Pretend that you remember that F is differentiable at $(x_0,y_0)\in\Omega$ if there is a linear function $L:\mathbf{R}^2\to\mathbf{R}^2$ such that

$$\lim_{(h,k)\to(0,0)}\frac{\|F(x_0+h,y_0+k)-F(x_0,y_0)-L(h,k)\|}{\|(h,k)\|}=0,$$

in which case, the partials of u and v must exist and L is given by the Jacobian Matrix

$$[L]=\begin{pmatrix}u_x(x_0,y_0) & u_y(x_0,y_0)\\ v_x(x_0,y_0) & v_y(x_0,y_0)\end{pmatrix}.$$

(Of course, here $\|(x,y)\|=\sqrt{x^2+y^2}=|x+iy|$.)

5. Let f, F, u, v and Ω be as above. Let $z_0=x_0+iy_0\in\Omega$. Show that $f'(z_0)$ exists if and only if the Cauchy-Riemann equations hold at z_0 and F is differentiable at (x_0,y_0). (Hint: if we let $z=h+ik$ and if T is given by the matrix

$$[T]=\begin{pmatrix}u_x(x_0,y_0) & -v_x(x_0,y_0)\\ v_x(x_0,y_0) & u_x(x_0,y_0)\end{pmatrix},$$

then

$$\|F(x_0+h,y_0+k)-F(x_0,y_0)-T(h,k)\|=|f(z+z_0)-f(z_0)-\omega z|,$$

where $\omega=u_x(x_0,y_0)+iv_x(x_0,y_0)=f_x(z_0)$. Then remember (1).)

Problem #5 has an important Corollary. We learn in multivariable calculus, that F is differentiable at (x_0,y_0) if the partial derivatives of u and v exist in a neighborhood of (x_0,y_0) and are continuous at (x_0,y_0). Hence we get as a Corollary of problem #5, with f, u and v defined as above, that if u and v have continuous partial derivatives in a neighborhood of (x_0,y_0) and if the Cauchy-Riemann equations hold at z_0, then $f'(z_0)$ exists. Use this observation in problem #6.

6. Define $\exp:\mathbf{C}\to\mathbf{C}$ by $\exp(x+iy)=e^x\big(\cos(y)+i\sin(y)\big)$. Show that $\exp\in H(\mathbf{C})$ and $\exp'(z)=\exp(z)$ for all $z\in\mathbf{C}$.