Math 73/103 Final Exam

Instructions: You should return your exam to me in my office between 2:00 and 3:00 on Monday afternoon, December 5, 2011. This is a closed book, closed notes exam.

1. (20) Recall that the *complex conjugate* of $z = x + iy$ is given by $\bar{z} := x - iy$. Determine all domains on which $f(z) = \bar{z}$ is holomorphic.

2. (25) Suppose that f has an isolated singularity at a and that the real part of f is bounded near a; that is, there is a $r > 0$ such that

$$|\operatorname{Re} f(z)| \leq M < \infty \quad \text{for all } z \in D_r'(a).$$

Show that a is a removable singularity for f. (Consider $g(z) = \exp\bigl(f(z)\bigr)$.)

3. (25) Suppose that Ω is a simply connected region and that $u : \Omega \to \mathbf{R}$ is harmonic. Show that u has a harmonic conjugate on Ω.

4. (25) Recall that we say f has a "pole at ∞" if $g(z) := f(\frac{1}{z})$ has a pole at 0. Show that an entire function has a pole at ∞ if and only if it is a polynomial.

5. (25) Suppose that f is entire and one-to-one. Show that $f(z) = az + b$ with $a \neq 0$. (Consider what types of singularities a one-to-one function can have at ∞. You can't invoke the full power of Picard's Theorem here, but you can use what we proved in class about about the local behavior of a function near an essential singularity.)

6. (30) Let μ, ν, and λ be σ-finite measures on $(X, \mathfrak{M})$. We'll denote the Radon-Nikodym derivative of ν by μ by $\dfrac{d\nu}{d\mu}$.

   (a) Show that if $\nu \ll \mu$ and $g : X \to [0, \infty]$ is measurable, then $\displaystyle\int_X g\, d\nu = \int_X g \frac{d\nu}{d\mu}\, d\mu$. (As observed in class, this is a Corollary of an old Theorem.) Conclude that $f \in L^1(\nu)$ if and only if $f \frac{d\nu}{d\mu} \in L^1(\mu)$, and that the same formula holds.

   (b) Suppose that $\nu \ll \mu \ll \lambda$. Show that $\dfrac{d\nu}{d\lambda} = \dfrac{d\nu}{d\mu} \dfrac{d\mu}{d\lambda}$. Of course, "=" means "equal almost everywhere $[\lambda]$."

(c) Suppose that $\mu \ll \nu$ and $\nu \ll \mu$ (we say the μ and ν are equivalent and write $\nu \approx \mu$). Show that $\dfrac{d\mu}{d\nu} = \left[\dfrac{d\nu}{d\mu}\right]^{-1}$. Again "=" means "equal almost everywhere $[\mu]$ (or $[\nu]$)".

7. (20) Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) = 1$. For each $n \in \mathbf{Z}_+$, let $A_n \in \mathfrak{M}$ be such that $\mu(A_n) = 1$. Show that if $A = \bigcap_n A_n$, then $\mu(A) = 1$.

8. (30) Suppose that $U := D_1(0)$ and $f \in H(U)$ is such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in U$.

(a) Let

$$F(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \text{ and} \\ f'(0) & \text{if } z = 0. \end{cases}$$

Show that $F \in H(U)$.

(b) Suppose that $\omega \in U$ and $0 < |\omega| < r < 1$. Let γ_r be the positively oriented circle of radius r centered at 0. Use the Maximum Modulus Principal to show that

$$|F(\omega)| \leq \max_{z \in \gamma_r^*} \frac{|f(z)|}{r} \leq \frac{1}{r}.$$

(c) Now prove that $|f(z)| \leq |z|$ for all $z \in U$ by letting $r \nearrow 1$ in the above. (This result is knows as Schwarz's Lemma.)