Math 71 Fall 2015
Homework #5: polynomial rings

(1) If A is a ring, denote by $A[X,Y]$ the ring of polynomials in two variables with coefficients in A.

  (a) Is the ideal (X,Y) of $\mathbb{Q}[X,Y]$ principal?

  (b) Are the ideals (X) and (X,Y) prime in $\mathbb{Q}[X,Y]$? Are they maximal?

  (c) Are the ideals (X,Y) and $(2,X,Y)$ prime in $\mathbb{Z}[X,Y]$? Are they maximal?

(2) Let F be a field and $A = F[X, X^2Y, X^3Y^2, \ldots, X^nY^{n-1}, \ldots] \subset F[X,Y]$.

  (a) Prove that A consists of all polynomials of the form $a_0 + \sum a_{k,\ell}X^kY^\ell$ with $a_{k,\ell} = 0$ if $1 \leq k \leq \ell$.

  (b) Prove that A and $F[X,Y]$ have the same field of fractions.

  (c) Prove that the ideal $(X, X^2Y, X^3Y^2, \ldots)$ of A is not finitely generated.

(3) Let $A = \mathbb{Z} + X\mathbb{Q}[X]$.

  (a) Prove that A is an integral domain and determine $A^\times$.

  (b) Show that the irreducibles of in A are prime numbers, opposites of prime numbers and irreducible polynomials in $\mathbb{Q}[X]$ with constant term ± 1.

  (c) Are the irreducibles prime?

  (d) Show that X cannot be written as the product of irreducibles. Is A a UFD?

(4) Let F be field and P a polynomial of degree $n \geq 1$.

  (a) Prove that $F[X]/(P)$ is a vector space of dimension n over F.

  (b) Assume that F is finite, of order q. What is the cardinality of $F[X]/(P)$?

  (c) Prove that $F[X]/(P)$ is a field if and only if P is irreducible.