Math 71 Fall 2015
Homework #2: cyclic groups and quotients

(1) Recall that an *automorphism* of a group is an isomorphism of the group with itself. The set of automorphisms of a group G is denoted by $\mathrm{Aut}(G)$.
Let G be a group, H a normal subgroup of G and K a subgroup of H.

(a) Assume that $\varphi(K) \subset K$ for any $\varphi \in \mathrm{Aut}(H)$. Is K normal in H?

(b) Under the same hypotheses, prove that K is normal in G.

(c) Find three groups G, H and K such that $K \lhd H$ and $H \lhd G$ but K is not normal in G.

(2) Recall that $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and consider the multiplicative groups

$$\mathbb{T} = \{z \in \mathbb{C}\ ,\ |z| = 1\}$$
$$\mu_n = \{z \in \mathbb{C}\ ,\ z^n = 1\}$$
$$\mu_\infty = \{z \in \mathbb{C}\ ,\ \exists n \in \mathbb{Z}\ ,\ z^n = 1\}.$$

(a) Which of these are cyclic?

(b) Prove the following isomorphisms:

$$\mathbb{R}/\mathbb{Z} \simeq \mathbb{T} \quad , \quad \mathbb{C}^\times/\mathbb{R}_+^\times \simeq \mathbb{T} \quad , \quad \mathbb{C}^\times/\mathbb{R}^\times \simeq \mathbb{T}$$
$$\mathbb{T}/\mu_n \simeq \mathbb{T} \quad , \quad \mathbb{C}^\times/\mu_n \simeq \mathbb{C}^\times \quad , \quad \mathbb{Q}/\mathbb{Z} \simeq \mu_\infty.$$

(c) Determine all the finite subgroups of μ_∞.

(3) The *center* of a group G is the set $Z(G) = \{\gamma \in G \mid \forall g \in G, \gamma \cdot g = g \cdot \gamma\}$.

(a) Compute the center of your favorite and least favorite non-abelian groups.

Let G be a group and H a subgroup such that $H \subset Z(G)$.

(b) Verify that $H \lhd G$.

(c) Prove that if G/H is cyclic then G is abelian.

(4) A *commutator* in a group G is an element of the form $xyx^{-1}y^{-1}$ with $x, y \in G$.

(a) Do commutators form a subgroup of G?

Let $D(G)$ be the subgroup of G generated by the commutators.

(b) Prove that $D(G)$ is normal in G.

(c) Prove that $G/D(G)$ is abelian.

(d) Prove that $D(G)$ is the smallest normal subgroup of G with abelian quotient.