Math 71 Fall 2015
Homework #1: groups, subgroups and morphisms

(1) Let E be the set of matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ with $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$.

(a) Prove that matrix multiplication is an associative composition law on E.

(b) Does E have a *left identity*, that is, an element e such that $e \cdot x = x$ for every $x \in E$? Is it unique? What about right identities?

(c) Does every element in E have a left inverse? A right inverse? Is it unique?

(2) Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; ; \; \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{matrix} \right\}$ and $\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Verify that G is a subgroup of $\mathrm{GL}(2, \mathbb{R})$ that contains α and β.

(b) Determine the order of α, β and $\alpha\beta$.

(3) All the sets considered below are equipped with their ordinary additive group structures; m and n are positive integers.

(a) Determine all the elements of $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$, $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z})$ and $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$

(b) Show that $\Gamma = \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is a group for a law to be determined.

(c) Prove that Γ is isomorphic to $\mathbb{Z}/(m \wedge n)\mathbb{Z}$.

(4) If A and B are subsets of a group G, we denote by AB the set of products $\{ab \, , \, a \in A \, , \, b \in B\}$. Let H_1 and H_2 be subgroups of G.

(a) Find a necessary and sufficient condition for $H_1 H_2$ to be a subgroup of G.

(b) Assume that H_1 and H_2 are finite and $H_1 \cap H_2 = \{e_G\}$. Prove that

$$\mathrm{Card}(H_1 H_2) = \mathrm{Card}(H_1)\,\mathrm{Card}(H_2).$$

Hint: construct a bijection between $H_1 \times H_2$ and $H_1 H_2$.