Math 71 Homework
Due Monday, October 24th.

- In addition to the already assigned problems 11, 18 and 19 on page 96:
- Page 101: 3 and
- Page 11: 14.
- And work the following problems:

1. Let n and m be positive integers. Recall that their *least common multiple*, written $[n, m]$, is the positive integer $l = [n, m]$ such that $n \mid l$, $m \mid l$ and such that if k is any other integer with $n \mid k$ and $m \mid k$, then $l \mid k$.

(a) Show that $[n, m]$ always exists.

(b) Show that if (n, m) is the greatest common divisor of n and m, then $(n, m)[n, m] = ab$.

(Please don't invoke the Fundamental Theorem of Arithmetic here. You can do these using techniques we've invoked in lectures such as the Well Ordering Property of $\mathbf{Z}$ and the division algorithm.)

2. Let n and m be positive integers. Recall that (n, m) is the greatest common divisor of n and m.

(a) Use the First Isomorphism Theorem to show that there is an injective homomorphism of $\mathbf{Z}/[n, m]\mathbf{Z}$ into the direct product $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$.

(b) Show that your map in part (a) is surjective (and therefore and isomorphism) if and only if $(n, m) = 1$. Conclude that is this case $\mathbf{Z}/nm\mathbf{Z} \cong \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$.

3. Let $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ be finite cyclic groups of orders n and m respectively. Show that $G_1 \times G_2$ is cyclic if and only if $(n, m) = 1$. Conclude that $\mathbf{Z}/nm\mathbf{Z} \cong \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ if and only if $(n, m) = 1$. (You've already done half of this in the previous problem.)