MATH6 - Introduction to Finite Mathematics

Exam I ANSWERS

May 1, 2007

1. (10 points) Calculate the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$.

Answer:

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] (R_1 + R_2 \to R_2) \longrightarrow \left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] (2R_1 + R_3 \to R_3)$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{array}\right] (-2R_2 + R_1 \to R_1) \longrightarrow \left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{array}\right] (-2R_2 + R_3 \to R_3)$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] (R_3 + R_2 \to R_2) \longrightarrow \left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] (-R_3 + R_1 \to R_1)$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right],$$

so

$$A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix}.$$

Douglas R. Hofstadter (1945 -): Hofstadter's Law: It always takes longer than you expect, even when you take into account Hofstadter's Law. (From *Gödel, Escher, Bach*, 1979.)

2. (12 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ be the encoding matrix for the following two problems. You may use the chart on the last page of the exam to aid your work.

(a) Encode the message: "Finite math is fun"

(b) Decode the message: Upon losing the use of his right eye, the famous mathematician Leonhard Euler (1707-1783) said: " Now I will have . . . " $\begin{bmatrix} 3 & 15 & 27 & 17 & 23 & 68 \\ -17 & -19 & -28 & -19 & -29 & -41 \\ -15 & -34 & -36 & -35 & -43 & -82 \end{bmatrix}$.

Answer:

(a)

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 9 & 0 & 20 & 9 & 6 & \\ 9 & 20 & 13 & 8 & 19 & 21 & 0 \\ 14 & 5 & 1 & 0 & 0 & 14 & \end{bmatrix} = \begin{bmatrix} 10 & 44 & 25 & 36 & 47 & 34 \\ -15 & -29 & -13 & -28 & -28 & -27 \\ -16 & -53 & -25 & -56 & -56 & -40 \end{bmatrix}.$$

(b) From problem number 1, we have

$$\begin{aligned} & A^{-1} \begin{bmatrix} 3 & 15 & 27 & 17 & 23 & 68 \\ -17 & -19 & -28 & -19 & -29 & -41 \\ -15 & -34 & -36 & -35 & -43 & -82 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 15 & 27 & 17 & 23 & 68 \\ -17 & -19 & -28 & -19 & -29 & -41 \\ -15 & -34 & -36 & -35 & -43 & -82 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 19 & 9 & 18 & 20 & 14 \\ 15 & 0 & 19 & 1 & 9 & 27 \\ 19 & 4 & 20 & 3 & 15 & 0 \end{bmatrix}, \end{aligned}$$

which gives that the quote is "Now I will have less distraction."

3. (10 points) ($*$) A search of available money market accounts yields the following offers:

(a) Republic Bank: 4.28% compounded continuously

(b) Chase Bank: 4.31% compounded daily

(c) BankFirst: 4.35% compounded monthly.

What is the APY of each?

Answer:

(a) $APY = e^r - 1 = e^{.0428} - 1.$

(b) $APY = (1 + \frac{r}{m})^m - 1 = (1 + \frac{.0431}{365})^{365} - 1.$

(c) $APY = (1 + \frac{r}{m})^m - 1 = (1 + \frac{.0435}{12})^{12} - 1.$

4. (12 points) A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers. The monthly cost of leasing each of these types of planes is $\$8,000$, $\$14,000$, and $\$16,000$.

(a) How many of each type of plane should be leased?

(b) Which of these solutions would minimize the monthly leasing cost?

Answer:

(a) Let T, F, and W denote the number of planes with 10, 15, and 20 passenger seats, respectively. Then we are solving the system

$$\begin{aligned} T \;+\; F + \; W &= 12 \\ 10T \;+\; 15F + \; 20W &= 220 \end{aligned}$$

Using augmented matrices, this means that the solution is given by

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{bmatrix} \quad (-10R_1 + R_2 \to R_2)$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{bmatrix} \quad \left(\frac{1}{5}R_2 \to R_2\right)$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{bmatrix} \quad (-R_2 + R_1 \to R_1)$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{bmatrix},$$

so if the corporation leases $W = t$ 20-passenger planes, then they must lease $F = 20 - 2t$ 15-passenger planes and $T = t - 8$ 10-passenger planes. Since each of these values must be a nonnegative integer, we have that the solution for (W, F, T) is one of: $(8, 4, 0), (9, 2, 1), (10, 0, 2)$.

(b) To minimize cost, we should minimize the monthly cost function, $8000(t-8)+14000(20-2t) + 16000t = 216000 - 4000t$, which is done when t is as large as possible. Putting $t = 10$ makes all of the variables nonnegative and no larger value of t does so, so the corporation should lease ten 20-passenger planes, no 15-passenger planes, and two 10-passenger planes.

5. (12 points) Describe the following systems of linear equations with as many of the following that apply: consistent, inconsistent, dependent or independent. Find the solution set where possible.

(a)

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 2x_1 + 4x_2 &= 5 \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 - 6x_2 &= 4 \\ 2x_1 - 3x_2 &= 2 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 5x_2 &= 1 \\ 2x_1 + 10x_2 &= 1 \end{aligned}$$

Answer:

(a) We have by the first equation that $2x_1 = 4 - 3x_2$. Substituting this into the second gives that $4 - 3x_2 + 4x_2 = 5$ or $x_2 = 1$. Then, the first equation gives that $x_1 = \frac{1}{2}$. Since this solution set is a single pair (x_1, x_2), the system is consistent and independent.

(b) This system is consistent, but dependent, as the first equation is twice the second equation. Thus the solution set is given by $x_1 = t$ and $x_2 = \frac{2t-2}{3}$.

(c) The coefficients on the left hand side of the second equation are twice those of the first, but the corresponding constants on the right hand side are the same, so this system is inconsistent.

6. (10 points) (∗) The West Coast Life Insurance Company offers a $300,000, 30-year term life insurance policy to its "super-preferred" customers at a rate of $21.96 a month. Assuming the value of money is compounded monthly at 6%, answer the following questions regarding a policy bought this year.

(a) What is the current (2007) total value of the payments made by a super-preferred customer over the course of the 30-year term?

(b) What is the total value of the payments at the end of the 30 years in 2037?

Answer:

(a)

$$PV = PMT\frac{1-(1+i)^{-n}}{i} = \$21.96\frac{1-1.005^{-30}}{.005}.$$

(b)

$$FV = PMT\frac{(1+i)^{n}-1}{i} = \$21.96\frac{1.005^{30}-1}{.005}.$$

Note: Due to the somewhat ambiguous original phrasing of this question, could also substitute $i = 0.06$, $PMT = 12(\$21.96)$, and $n = 30$ into both parts.

7. (10 points) The People's Republic of Thalia has two primary food sources: pizza and aardvarks. These two industries depend on each other. For example, to produce a $10 pizza, the requirements are $4 of aardvark meat (for the pizza) and $4 of pizza (for the cooks). To produce $50 of aardvark meat requires $10 of aardvark meat (for feed) and a $10 pizza (for the butchers).

(a) What final output is required to meet a demand of $10M worth of pizza and $20M dollars worth of aardvark meat?

(b) What output is required to meet a demand of $20M worth of pizza and $10M dollars worth of aardvark meat?

Answer:

The technology matrix is given by:

$$\begin{array}{cc} & \quad\textit{Output} \\ & \quad A \quad P \\ \textit{Input} \begin{array}{c} A \\ P \end{array} & \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \end{array} = M,$$

and so the solution is given by the formula

$$X = (I - M)^{-1} D = \begin{bmatrix} 0.8 & -0.4 \\ -0.2 & 0.6 \end{bmatrix}^{-1} D = \frac{1}{0.4} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} D = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 2 \end{bmatrix} D,$$

which, for each value of D, is equal to:

(a)

$$\begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} \$20M \\ \$10M \end{bmatrix} = \begin{bmatrix} \$40M \\ \$30M \end{bmatrix} \begin{array}{c} A \\ P \end{array}$$

(b)

$$\begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} \$20M \\ \$10M \end{bmatrix} = \begin{bmatrix} \$35M \\ \$45M \end{bmatrix} \begin{array}{c} A \\ P \end{array}$$

8. (12 points) (*) You have come upon a potentially lucrative idea for a new business. To afford start-up costs, you borrow $\$125,000$ at 9% compounded monthly for ten years.

(a) What is the monthly payment?

(b) What is the unpaid balance at the end of the first year?

(c) How much interest is paid during the first year?

Answer:

(a)

$$PMT = PV\frac{i}{1-(1+i)^{-n}} = \$125000\frac{.0075}{1-1.0075^{-120}}.$$

(b) The unpaid balance is given by the present value one year after the loan began, i.e.

$$PMT\frac{1-1.0075^{-108}}{.0075} = \$125000\frac{1-1.0075^{-108}}{1-1.0075^{-120}}.$$

(c) The interest is given by the difference between the payments made over the first year and the amount which went to principal [the loan amount less the amount in part b)], i.e.

$$(12)(\$125000)\frac{.0075}{1-1.0075^{-120}} - \left(\$125000 - \$125000\frac{1-1.0075^{-108}}{1-1.0075^{-120}}\right).$$

9. (12 points) Let A, B, and C be arbitrarily chosen invertible $n \times n$ matrices, and let O be the $n \times n$ zero matrix. You need not show any work for this problem.

True or False:

$$A + (B + C) = (A + B) + C \quad \_\_\_\_\_\_$$

$$(AB)C = B(AC) \quad \_\_\_\_\_\_$$

$$A(B + C) = BA + BC \quad \_\_\_\_\_\_$$

$$(A + B)C = AC + BC \quad \_\_\_\_\_\_$$

$$\text{If } ABA^{-1} = B\text{, then either } A = I_n \text{ or } B = I_n \quad \_\_\_\_\_\_$$

$$(A + A^{-1})(I_n + B - AA^{-1} - B) = O \quad \_\_\_\_\_\_$$

Answer:

1. True, by the associative property.
2. False, the associative property gives $(AB)C = A(BC)$, but the commutative property does not necessarily hold for A and B.
3. False, the distributive property gives that $A(B+C) = AB+AC$, but the commutative property does not necessarily hold for A and B or C.
4. True, by the distributive property.
5. False, $A = B = 2I_n$ is one counterexample.
6. True, $I_n + B - AA^{-1} - B = I_n + B - I_n - B = O$ and $(A + A^{-1})O = O$.

Robert D. Carmichael (1879-1967): "A thing is obvious mathematically after you see it."
In N. Rose (ed.) *Mathematical Maxims and Minims*, Raleigh NC: Rome Press Inc., 1988.

Eric Temple Bell (1883-1960): "Obvious is the most dangerous word in mathematics."

Letter	Number
Blank	0
A	1
B	2
C	3
D	4
E	5
F	6
G	7
H	8
I	9
J	10
K	11
L	12
M	13
N	14
O	15
P	16
Q	17
R	18
S	19
T	20
U	21
V	22
W	23
X	24
Y	25
Z	26
.	27
!	28
?	29