MATH6 - Introduction to Finite Mathematics

Final Exam ANSWERS

June 2, 2007

1. (16 points) Find the solution set for the following system:

$$\begin{aligned} 3x- \;\; 2y- \;\; 8z+ \;\; 7t &= -1 \\ x+ \;\; y- \;\; z- \;\; t &= 3 \\ x- \;\; y- \;\; 3z+ \;\; 3t &= -1. \end{aligned}$$

Answer:

$$\left[\begin{array}{cccc|c} 3 & -2 & -8 & 7 & -1 \\ 1 & 1 & -1 & -1 & 3 \\ 1 & -1 & -3 & 3 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 3 \\ 3 & -2 & -8 & 7 & -1 \\ 1 & -1 & -3 & 3 & -1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 3 \\ 0 & -5 & -5 & 10 & -10 \\ 1 & -1 & -3 & 3 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 3 \\ 0 & -5 & -5 & 10 & -10 \\ 0 & -2 & -2 & 4 & -4 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 3 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & -5 & -5 & 10 & -10 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 3 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, if we put $z = s_1$ and $t = s_2$, then $y = 2 + 2s_2 - s_1$ and $x = 1 - s_2 + 2s_1$. Summing up, we have the solution set is

$$\{(x, y, z, t) : (x, y) = (1 - t + 2z, 2 + 2t - z)\}.$$

Paul Erdős (1913 - 1996): A mathematician is a machine for turning coffee into theorems.

2. (14 points) Let P be the set of prime numbers.

(a) True or False:

(i) $2 \in P$.

(ii) $\{2\} \in P$.

(iii) $2 \subseteq P$.

(iv) $(\{2\} \cap P) \subseteq P$.

(v) $(\{0\} \cup P) \subseteq P$.

(b) The following two questions refer to this statement: If it is cloudy, then it will rain.

(i) What is the contrapositive of this statement?

(ii) What is the converse of this statement?

Answer:

(a) (i)True

(ii)False

(iii)False

(iv)True

(v)False

(b) (i)If it will not rain, then it is not cloudy.

(ii)If it will rain, then it is cloudy.

3. (15 points) ($*$) How many five-letter "words" can you make if a "word" is not allowed to have two consecutive consonants or two consecutive vowels? (For the purpose of this question, consider a y as a vowel.)

Answer:

Since we must alternate between consonants and vowels in any given "word", we can either have three consonants and two vowels or three vowels and two consonants. Thus the total number of words is

$$\begin{aligned} 20^3 \cdot 6^2 + 20^2 \cdot 6^3 &= 20^2 \cdot 6^2(20+6) \\ &= 120^2 \cdot 26 \\ &= 374400. \end{aligned}$$

4. (15 points) A ball is drawn from an urn that contains 6 balls numbered 1-6. 1-3 are red, 4 is green, and 5-6 are blue. Let R and B be the events that the ball is red and blue, respectively, and let O be the event that the number on the ball is odd. Are any two of these events independent?

Answer:

We have the following probabilities:

$P(R)$	$\frac{1}{2}$
$P(B)$	$\frac{1}{3}$
$P(R \cap B)$	0
$P(O)$	$\frac{1}{2}$
$P(R \cap O)$	$\frac{1}{3}$
$P(B \cap O)$	$\frac{1}{6}$

Testing the three possible pairs for independence, we see:

$$\begin{aligned} P(R)P(B) &= \frac{1}{6} & \neq 0 &= P(R \cap B) \\ P(R)P(O) &= \frac{1}{4} & \neq \frac{1}{3} &= P(R \cap O) \\ P(B)P(O) &= \frac{1}{6} & &= P(B \cap O), \end{aligned}$$

which means that B and O are independent events, but neither B and R nor R and O are independent of one another.

5. (25 points) Suppose you are solving a two-industry Leontief Input-Output problem, and you determine that the technology matrix is

$$\begin{array}{cc} & \textit{Output} \\ & \begin{array}{cc} A & B \end{array} \\ \textit{Input} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix} \end{array} = M,$$

and the demand matrix is

$$\begin{array}{c} A \\ B \end{array} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = D.$$

What is the output matrix necessary to meet the demand in this model?

Answer:

Note: Due to a mix-up in the printing of the exam, the formula for getting the answer from a calculator was given full credit. Had the numbers been correct, the solution is as follows.

Recall that the solution is given by the formula

$$\begin{aligned} X &= (I - M)^{-1} D \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 20 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 10 \end{bmatrix} \\ &= \frac{1}{0.64 - 0.14} \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.8 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} \\ &= 2 \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.8 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 36 \\ 44 \end{bmatrix}. \end{aligned}$$

6. (20 points)

(a) Given the augmented matrices below for systems of linear equations in the variables x, y, and z, what is the solution set of each system?

$$\text{(i)} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \text{(ii)} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\text{(iii)} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{(iv)} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(b) Show that the matrix

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

has no inverse.

Answer:

(a) (i) $(x, y, z) = (-1, -1, -1)$

(ii) $(x, y, z) = (3, 1, 2)$

(iii) $(x, y, z) = (2 - t, t, -1)$ for any choice of t

(iv) no solution

(b) Assume to the contrary that the matrix does have an inverse, and put

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, by the definition of inverse, we have that $3a + 4c = 1$ and $6a + 8c = 0$. Dividing this last equation by 2, we have that $3a + 4c = 1$ and $3a + 4c = 0$, which cannot be simultaneously true. Thus the inverse does not exist.

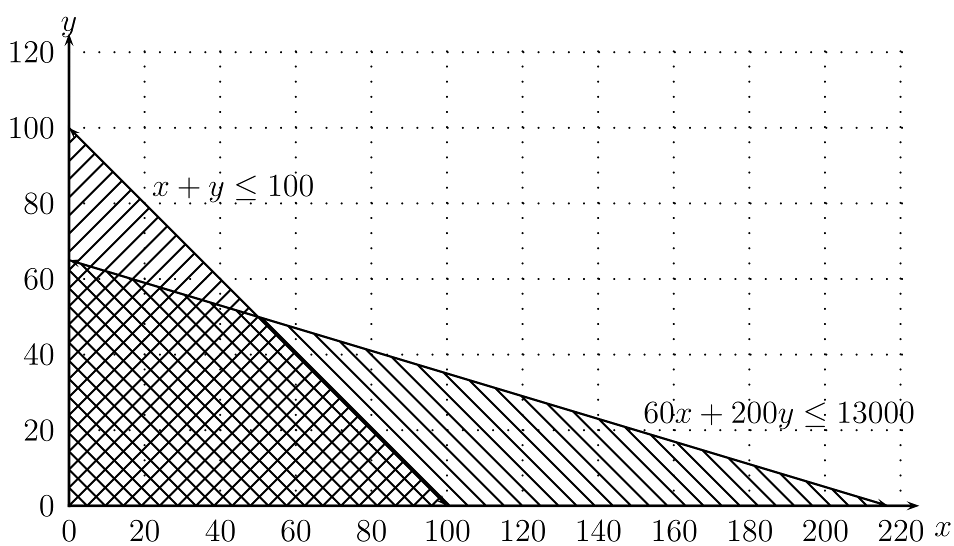
7. (25 points) An electronics manufacturing company produces two kinds of TV's: standard and hi-definition. It costs \$60 to make a standard TV which sells for \$100. It costs \$200 to make a hi-definition TV which sells for \$350. The daily production capacity is 100 TV's, and the daily operating cost may not exceed \$13,000. How many TV's of each kind should the company produce to maximize its profit, and what is the maximum profit value?

Answer:

Letting x represent the number of standard TV's and y the number of hi-definition TV's the company produces, we have the following linear programming problem:

$$\begin{aligned} \text{maximize:} \quad & 40x + 150y \\ \text{subject to:} \quad & x + y \leq 100 \\ & 60x + 200y \leq 13,000 \\ & x, y \geq 0. \end{aligned}$$

Plotting, we see that the feasible region looks like:



Testing the corner points:

Point	Value
(100,0)	\$4,000
(0,65)	\$9,750
(50,50)	\$9,500

Thus, we see that the maximum profit of \$9,750 is made when the company produces 65 hi-definition TV's and no standard TV's.

8. (15 points) (∗) Your friend wants to buy a new car priced at \$25,000. She can choose between the dealer's offer of 0% financing for 50 months with no rebate or a \$5,000 dealer's rebate and pay the remainder at 6.0% financing for 50 months through a local bank. Which would you suggest and why? (You need only use formulas here to explain how you would make a decision if a calculator were available.)

Answer:

Considering the dealer's offer of 0% financing, we have the payments are

$$\frac{\$25,000}{50} = \$500. \tag{1}$$

If she instead opts for the rebate and uses the bank's financing, we can solve the formula

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

for the payment (PMT) to find that the payment is

$$\begin{aligned} PMT &= PV\frac{i}{1-(1+i)^{-n}} \\ &= \$20,000\frac{.005}{1-(1.005)^{-50}} \\ &= \frac{\$100}{1-(1.005)^{-50}}. \end{aligned} \tag{2}$$

To make a sound financial decision, your friend should compare the payments in (1) and (2) and opt for the smallest of the two.

9. (20 points) Three cards are drawn at random without replacement from a standard deck, and the color (either black or red) is noted. (Recall there are 26 black cards and 26 red cards in a standard deck.)

(a) Draw a tree diagram for this situation.

(b) Find the probability of all of the possible outcomes.

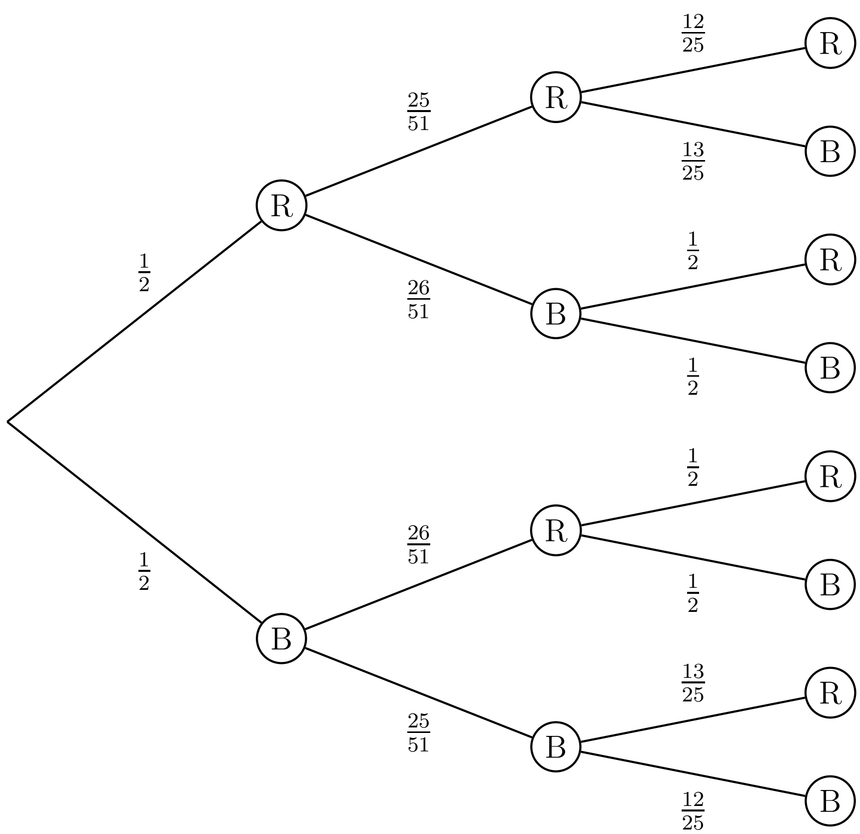
(c) What is the probability that all three cards are red?

(d) What is the probability that the second card is red?

(e) What is the probability that the third card is red given that the second is red?

Answer:

(a)



Jason Kidd (1973 -): (upon being drafted by the Dallas Mavericks in 1994) We're going to turn this team around 360 degrees.

(b)

Sequence	Probability
RRR	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{24}{51} = \frac{2}{17}$
RRB	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
RBR	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
RBB	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
BRR	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
BRB	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
BBR	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{102}$
BBB	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{24}{51} = \frac{2}{17}$

(c) Reading from our table, $P(RRR) = \frac{2}{17}$.

(d) Adding the probabilities of those in the table with an R in the middle, we have

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{24}{51} + \frac{26}{51} + \frac{26}{51} + \frac{26}{51}\right) = \frac{1}{4} \cdot \frac{102}{51} = \frac{1}{2}.$$

(e) Using Bayes' Theorem, we find the probability is

$$\frac{\frac{2}{17} + \frac{13}{102}}{\frac{1}{2}} = \frac{\frac{25}{102}}{\frac{1}{2}} = \frac{25}{51}.$$

10. (35 points) A voting district can elect three possible representatives to the U.S. House of Representatives: a Democrat (D), Independent (I), or Republican (R). An incumbent representative has a 60% chance of re-election if he or she is a Democrat or Republican and a 40% chance or re-election if he or she is an Independent. A Democratic challenger has a 40% chance of unseating an independent incumbent and a 30% chance of winning election against a Republican incumbent. An Independent has a 20% chance of being elected over a Democratic incumbent.

(a) Draw a transition diagram to represent this situation.

(b) Give a transition matrix that represents this situation.

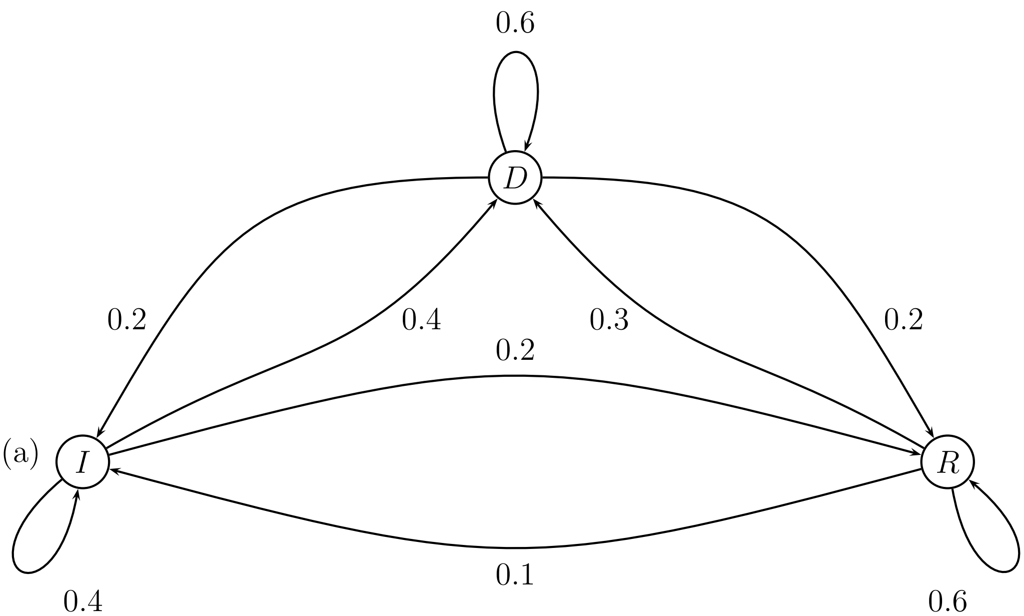
(c) If the district currently has an Independent representative, what are the probabilities that it will have a Democrat, Independent, or Republican in the next election? Two elections from now?

(d) What kind of Markov Chain is this? What does this say about the probability of this district's representative being a Democrat, Independent, or Republican in the long-run? (Describe how you would answer this question if you had a calculator available, and cite any applicable theorems.)

(e) How would your answer to (d) change if the district currently has a Democratic representative?

Answer:

(a)



(b) The transition matrix is given by

$$P = \begin{array}{c} \\ D \\ I \\ R \end{array} \begin{array}{c} \begin{array}{ccc} D & I & R \end{array} \\ \begin{bmatrix} .6 & .2 & .2 \\ .4 & .4 & .2 \\ .3 & .1 & .6 \end{bmatrix} \end{array}$$

(c) Here, we have $S_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, so after the first election, the probabilities of a Democrat, Independent, and Republican are listed in order in $S_1 = \begin{bmatrix} .4 & .4 & .2 \end{bmatrix}$. After the second election, the probabilities are in the second state matrix

$$S_2 = S_1 P = \begin{bmatrix} .4 & .4 & .2 \end{bmatrix} \begin{bmatrix} .6 & .2 & .2 \\ .4 & .4 & .2 \\ .3 & .1 & .6 \end{bmatrix} = \begin{bmatrix} .46 & .26 & .28 \end{bmatrix}.$$

(d) This chain is a regular Markov Chain since its transition matrix has a power (the first power) with all positive entries. This means that it has a limiting matrix $\overline{P}$ and a unique stationary matrix S which is approached in the long-run by any initial state matrix S_0. Further, if $S = \begin{bmatrix} s_D & s_I & s_R \end{bmatrix}$, we have

$$\overline{P} = \begin{bmatrix} s_D & s_I & s_R \\ s_D & s_I & s_R \\ s_D & s_I & s_R \end{bmatrix}.$$

Thus the long run probability of moving from state i to state j is s_j for $j = D$, I, or R regardless of which of the three states i corresponds to.

(e) Since the stationary matrix is unique and is approached from any initial state matrix, starting with this different initial state matrix will not affect the long-run answer in (d).