Lecture 5: Examples in Permutations and Combinations

We define the numbers $P(n,k)$ and $\binom{n}{k}$.

$P(n,k)$ is the number of length k permutations from an n-element set.

$\binom{n}{k}$ is the number of k-element subsets of an n-element set.

There are algebraic formulas for these numbers, but these are the definitions.

Here are the permutations of $\{1,2,3,4\}$

1234	2134	3412	4312
1243	2143	3421	4321
1324	3124	2413	4213
1342	3142	2431	4231
1423	4123	2314	3214
1432	4132	2341	3241

We can picture these better with a deck of cards. Pick four cards from your deck, and keep them.

Supposing we pick A♡, 2♧, 3♤, 4◇, we can assign the numbers 1, 2, 3, and 4 to the cards in that order, so that each ordering of the cards corresponds exactly to one of the $4! = 24$ permutations on the board.

What if I wanted to look at length 2 sequences from this set of cards? The number of such is $P(4,2)$ by its definition, but how would I calculate this?

I can get a permutation of 4 elements by first taking a length 2 ~~pe~~ sequence from those elements, and then arranging the remaining cards. We can think of this as grouping the permutations so that the first 2 entries are the same. We get the following groupings:

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} 3 & 4 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 4 & 3 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 & 2 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 4 & 1 & 3 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 4 & 2 & 1 & 3 \\ 4 & 2 & 3 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 & 1 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 2 & 1 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

Each grouping has $2!$ elements.

We get that $P(4,2)\,2! = 4!$

We can generalize this to the following theorem:

<u>Thm</u>: $P(n,k)(n-k)! = n!$

<u>Pf</u>: Pick a length k sequence from your n-element set, then arrange the remaining $n-k$ elements. There are $P(n,k)$ sequences and $(n-k)!$ ways to arrange the remaining elements. What you get is a full permutation on n-elements, of which there are $n!$.

We can look at combinations analogously.

Looking at all the permutations of $\{1,2,3,4\}$, or all orderings ~~af~~ of our four cards, we group together those that have the first two cards the same, in any order.

1 2 3 4	2 1 3 4	3 4 1 2	4 3 1 2
1 2 4 3	2 1 4 3	3 4 2 1	4 3 2 1
1 3 2 4	3 1 2 4	2 4 1 3	4 2 1 3
1 3 4 2	3 1 4 2	2 4 3 1	4 2 3 1
1 4 2 3	4 1 2 3	2 3 1 4	3 2 1 4
1 4 3 2	4 1 3 2	2 3 4 1	3 2 4 1

The number of groupings is then $\binom{4}{2}$, by definition. To get the number in each grouping we first order our pair ($2!$ ways) then order the remaining elements ($2!$ ways)

So we get : $\binom{4}{2} 2!\, 2! = 4!$

Again, we can generalize to a theorem:

Thm: $\binom{n}{k} k!\,(n-k)! = n!$

Pf: Pick which k elements will be the first k ($\binom{n}{k}$ ways). Order them (since the k elements have already been chosen, there are $k!$ ways to order them). Order the remaining $n-k$ elements ($(n-k)!$ ways). This yields a permutation of all n elements ($n!$ ways).

The algebraic formulas for $P(n,k)$ and $\binom{n}{k}$ are simply manipulations from these two theorems.

For probability, the number of successful ~~et~~ outcomes and the total number of outcomes needs to be counted.

Let E be the event that the two (2♠) is in our chosen pair.

$$p(E) = \frac{n(E)}{n(S)}.$$

$n(S) = \binom{4}{2}$ the total number of pairs. $n(E) = 1 \cdot \binom{3}{1}$, since the 2♣ must be chosen and there are 3 choices for the remaining card.

Thus, $p(E) = \frac{3}{6} = \frac{1}{2}$

This is the general method for finding probability in these cases.