Math 69
Winter 2017
Friday, January 6

Truth Assignments and Tautological Implication

If v is a truth assignment on the set of sentence symbols (v is a function that assigns each sentence symbol to T or F), we extend v to a truth assignment $\overline{v}$ on all formulas by recursion on formulas:

$$\overline{v}(A_n) = v(A_n)$$

$$\overline{v}((\neg\alpha)) = \begin{cases} T & \overline{v}(\alpha) = F \\ F & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \wedge \beta)) = \begin{cases} T & \overline{v}(\alpha) = \overline{v}(\beta) = T \\ F & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \vee \beta)) = \begin{cases} F & \overline{v}(\alpha) = \overline{v}(\beta) = F \\ T & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \to \beta)) = \begin{cases} F & \overline{v}(\alpha) = T \;\&\; \overline{v}(\beta) = F \\ T & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \leftrightarrow \beta)) = \begin{cases} T & \overline{v}(\alpha) = \overline{v}(\beta) \\ F & \text{otherwise.} \end{cases}$$

Another way to phrase this is using Boolean functions as discussed in Wednesday's handout. That is, we can define

$$Val_{\neg}(X) = \begin{cases} T & X = F \\ F & X = T \end{cases}$$

$$Val_{\wedge}(X, Y) = \begin{cases} T & X = Y = T \\ F & \text{otherwise.} \end{cases}$$

$$Val_{\vee}(X, Y) = \begin{cases} F & X = Y = F \\ T & \text{otherwise.} \end{cases}$$

$$Val_{\rightarrow}(X, Y) = \begin{cases} F & X = T \,\&\, Y = F \\ T & \text{otherwise.} \end{cases}$$

$$Val_{\leftrightarrow}(X, Y) = \begin{cases} T & X = Y \\ F & \text{otherwise.} \end{cases}$$

Then we can define $\overline{v}$ by

$$\overline{v}(A_n) = v(A_n)$$

$$\overline{v}((\neg\alpha)) = Val_{\neg}(\overline{v}(\alpha)),$$

and for any binary connective $*$,

$$\overline{v}((\alpha * \beta)) = Val_{*}(\overline{v}(\alpha), \overline{v}(\beta)).$$

This notation might simplify the following task.

Prove the following proposition, which will be useful later in this handout, by induction[1] on formulas:

Proposition: For every two truth assignments v and w that agree with each other on every sentence symbol that occurs in α, we have $\overline{v}(\alpha) = \overline{w}(\alpha)$.

[1] At the end of this handout is an example of a proof by induction on formulas.

Here are some key definitions from the reading:

A truth assignment v *satisfies* a wff α if $\overline{v}(\alpha) = T$.

A truth assignment v *satisfies* a set Σ of wffs if v satisfies every wff in Σ.

A set Σ of wffs is *satisfiable* if there is a truth assignment that satisfies Σ.

If Σ is a set of wffs and α is a wff, then Σ *tautologically implies* α, written

$$\Sigma \models \alpha,$$

if every truth assignment v that satisfies Σ also satisfies α. Informally, whenever every wff in Σ is true, then α is also true.

If α and β are wffs, we say α *tautologically implies* β, written

$$\alpha \models \beta$$

if $\{\alpha\}$ tautologically implies β. That is, α tautologically implies β if every truth assignment that satisfies α also satisfies β.

Wffs α and β are *tautologically equivalent* if each tautologically implies the other. That is, α and β are tautologically equivalent if for every truth assignment v, we have $\overline{v}(\alpha) = \overline{v}(\beta)$.

Show that the following are tautologically equivalent:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta$$

$$\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))$$

Of course, neither of the above is actually a formula. We will eliminate parentheses when that can be done unambiguously; the textbook gives rules for eliminating parentheses at the end of section 1.3. Officially, the formula $(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta$ is an abbreviation for the actual wff

$$((\alpha_1 \wedge (\alpha_2 \wedge (\cdots \wedge \alpha_n) \cdots)) \to \beta).$$

Avoid omitting parentheses in formulas involving $\to$ and $\leftrightarrow$. Officially, for example, $A \leftrightarrow B \leftrightarrow C$ is an abbreviation for $(A \leftrightarrow (B \leftrightarrow C))$, and does NOT mean that A, B, and C have the same truth value. The textbook may abbreviate the formula $(\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots)))$ as

$$\alpha_n \to \alpha_{n-1} \to \cdots \to \alpha_1 \to \beta.$$

I suggest avoiding this.

Show the following:

$\Sigma \models \alpha$ if and only if $\Sigma \cup \{\neg\alpha\}$ is not satisfiable.

If Σ is satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is satisfiable.

A set of formulas Σ is said to be *finitely satisfiable* if every finite subset of Σ is satisfiable. We are about to prove the Compactness Theorem: If Σ is finitely satisfiable, then Σ is satisfiable. Prove the following proposition, which we will use as a lemma:

Proposition: If Σ is finitely satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.

Here is an outline of the proof of the Compactness Theorem. Fill in the missing details.

Suppose that Σ is finitely satisfiable. We must show that Σ is satisfiable. Define, by induction on n,

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\} & \text{if this is finitely satisfiable;} \\ \Sigma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Show that each Σ_n is finitely satisfiable.

Now let $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$. Show that Σ^* is finitely satisfiable.

Note that $\Sigma \subseteq \Sigma^*$, and that for each n, either A_n or $\neg A_n$ is in Σ^* (but not both). Define a truth assignment v by

$$v(A_n) = \begin{cases} T & A_n \in \Sigma^* \\ F & \neg A_n \in \Sigma^* \end{cases}$$

Show that v satisfies Σ^*, and therefore Σ (showing that Σ is satisfiable), as follows:

Suppose not. Let $\alpha \in \Sigma^*$ with $\overline{v}(\alpha) = F$. For each sentence symbol A_n, define

$$\beta_n = \begin{cases} A_n & A_n \in \Sigma^* \\ \neg A_n & \neg A_n \in \Sigma^* \end{cases}$$

Let Γ be the finite subset of Σ^* defined by

$$\Gamma = \{\alpha\} \cup \{\beta_n \mid A_n \text{ occurs in } \alpha\}.$$

Because Σ^* is finitely satisfiable, there is a truth assignment w satisfying Γ. Deduce a contradiction.

Here is an example of a proof by induction on formulas. It is a more careful version of an argument in the textbook.

Proposition: Every formula has balanced parentheses. (This means it has the same number of left parenthesis symbols as right parenthesis symbols.)

Proof: Prove this by induction on formulas.

Base Case: Suppose α is a sentence symbol, and show α has balanced parentheses. The formula α has 0 left parentheses and 0 right parentheses. Therefore α has balanced parentheses.

Inductive Step for $\neg$: Assume that α has balanced parentheses, and show that $(\neg\alpha)$ has balanced parentheses.

We know α has balanced parentheses; say α has m left parentheses and m right parentheses. The formula $(\neg\alpha)$ has these same parentheses, plus one more left parenthesis at the beginning and one more right parenthesis at the end. That is, $(\neg\alpha)$ has $m+1$ left parentheses and $m+1$ right parentheses. Therefore $(\neg\alpha)$ has balanced parentheses.

Inductive Steps for Binary Connectives: Assume that α and β have balanced parentheses, and that $*$ is a binary connective ($\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$), and show that $(\alpha * \beta)$ has balanced parentheses.

Say α has m left parentheses and m right parentheses, and β has n left parentheses and n right parentheses. The formula $(\alpha * \beta)$ has the parentheses from α and β, plus one more left parenthesis at the beginning and one more right parenthesis at the end. That is, $(\alpha * \beta)$ has $m+n+1$ left parentheses and $m+n+1$ right parentheses. Therefore $(\alpha * \beta)$ has balanced parentheses.

This completes the proof.