Math 69
Winter 2017
Friday, February 3

Proposition: $\Sigma \vdash \alpha$ iff $\Sigma \cup \Lambda \models_{taut} \alpha$.

Rule T: If $\Sigma \vdash \alpha_1$, ..., $\Sigma \vdash \alpha_n$, and $\{\alpha_1, \ldots, \alpha_n\} \models_{taut} \beta$, then $\Sigma \vdash \beta$.

Deduction Theorem: If $\Sigma \cup \{\alpha\} \vdash \beta$ then $\Sigma \vdash (\alpha \to \beta)$.

Contraposition: $\Sigma \cup \{\neg\alpha\} \vdash \beta$ iff $\Sigma \cup \{\neg\beta\} \vdash \alpha$.

Reduction ad Absurdum: If $\Sigma \cup \{\alpha\}$ is inconsistent, then $\Sigma \vdash \neg\alpha$.

Generalization: If y does not occur free in Σ, and $\Sigma \vdash \alpha$, then $\Sigma \vdash \forall y \alpha$.

Generalization on Constants: If c is a constant symbol that does not occur in Σ, and $\Sigma \vdash \alpha$ via a deduction in which the variable y does not occur, then $\Sigma \vdash \forall y\, \alpha^c_y$ via a deduction in which c does not occur.

Example: Show
$$\vdash \alpha_c^x \to \exists x\, \alpha.$$

Example: Show
$$\vdash \forall x \forall y(x = y \to y = x).$$

Exercise 1: Show

$$\vdash (\exists x\alpha \vee \exists x\beta) \leftrightarrow \exists x(\alpha \vee \beta).$$

Hint to get started: This wff is tautologically equivalent to a set of two wffs, each of the form $\theta \to \psi$. By Rule T, it suffices to show each of these wffs is deducible.

Exercise 2: Show that if the constant symbol c does not occur in α then $\{\exists x\, \alpha \to \alpha^x_c\}$ is consistent. You may want to use the Re-replacement Lemma from homework:

> If y is a variable that does not occur in α, then (y is substitutable for x in α and) x is substitutable for y in α^x_y, and $(\alpha^x_y)^y_x = \alpha$.

as well as the following related fact:

> If y is a variable that does not occur in α, and c a constant symbol that does not occur in α, then $(\alpha^x_y)^y_c = \alpha^x_c$ and $(\alpha^x_c)^c_y = \alpha^x_y$.

You may assume that $\emptyset$ is consistent.

Hint: Try to show that if $\{\exists x\, \alpha \to \alpha^x_c\}$ is inconsistent, then $\emptyset$ is inconsistent. At some point you will need to use Generalization on Constants.

Exercise 3: Extend exercise 2: Show that if the constant symbol c does not occur in α or Σ, and Σ is consistent, then $\Sigma \cup \{\exists x\, \alpha \to \alpha^x_c\}$ is consistent.