Math 69
Winter 2017
Wednesday, February 1

A *deduction* of a wff α from a set of wffs Σ is a finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of wffs such that each α_i is one of:

1. in Λ (a logical axiom);
2. in Σ (a hypothesis); or
3. derived from earlier formulas by modus ponens;

and $\alpha_n = \alpha$.

The set Λ of logical axioms contains all generalizations of formulas of the following forms:

1. Tautologies;
2. $\forall x\alpha \to \alpha^x_t$ where t is substitutable for x in α;
3. $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$;
4. $\alpha \to \forall x\alpha$ where x does not occur free in α;
5. $x = x$;
6. $x = y \to (\alpha \to \alpha')$ where α is atomic, and α' is obtained from α by replacing some, but not necessarily all, occurrences of x with y.

Recall that t is substitutable for x in α means that if a variable v occurs in t, then in α there is no free occurrence of x in the scope of any quantifier $\forall v$ or $\exists v$. For example, in the language of arithmetic, Sy is not substitutable for x in the formula $\exists y(x < y)$, because the variable y occurs in Sy, and there is a free occurrence of x in the scope of the quantifier $\exists y$. This is good, because we do not want

$$\forall x\exists y(x < y) \to \exists y(Sy < y)$$

to be a logical axiom.

Exercise: Explain why this is not a logical axiom in Group 6:

$$v = w \to (Pwz \to Pvz).$$

Explain why this is a logical axiom in Group 1:

$$\forall x(Px \to \neg\neg Px);$$

but this is not:

$$\forall x Px \to \forall x \neg\neg Px.$$

Identify which of the following are logical axioms in Group 2, where c is a constant symbol, f is a one-place function symbol, P is a two-place predicate symbol, and u, v, w, x, y, z are variables:

$$\forall u \exists v \neg Puv \to \exists v \neg Pcv$$

$$\forall y Pzy \to Pzfz$$

$$\forall x \exists y Pyx \to \exists y Pyfx$$

$$\forall x \exists y Pyx \to \exists y Pyfy$$

$$\forall x Pxx \to Pxx$$

$$\forall x \exists y Pyx \to \exists y Pyffz$$

$$\forall x \exists y Pyx \to \exists y Pfyx$$

Exercise: Give a deduction (from $\emptyset$) of $\forall x P x \to \forall x \neg\neg P x$. (This is not a logical axiom, but it has a three-line deduction.)

Exercise: Show

$$\{\forall x(Px \to Qx),\, \forall z Pz\} \vdash Qc$$

(where c is a constant symbol) by giving a deduction.

Show

$$\{\forall x(Px \to Qx),\, \forall z Pz\} \vdash Qy$$

(where y is a variable symbol) by giving a deduction.

Exercise: It is not hard to see that the set Λ of logical axioms is a decidable set. Explain why, if Σ is any decidable set of formulas, then

$$\{\alpha \mid \Sigma \vdash \alpha\}$$

is effectively enumerable.

Exercise: Show that for any set of wffs Σ and wff α,

$$\Sigma \vdash \alpha \iff \Sigma \cup \Lambda \models_{taut} \alpha$$

(where $\models_{taut}$ denotes tautological implication).

Conclude that if $\Sigma \vdash \alpha_1$, $\Sigma \vdash \alpha_2$, …, $\Sigma \vdash \alpha_n$, and

$$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \models_{taut} \beta$$

then $\Sigma \vdash \beta$. (The textbook calls this result Rule T. It is useful if you want to demonstrate that $\Sigma \vdash \beta$ without giving an actual deduction of β from Σ.)

Exercise: It is not hard to see that that

$$v = w \to (Pwz \to Pvz)$$

is a valid formula, where P is a two-place predicate symbol, and v, w, and z are variables. Note that this is not a logical axiom.

Give a deduction of this formula.