Ramsey's Theorem and Compactness

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Abstract

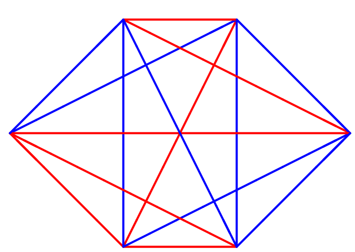
This paper is an example of an expository mathematics paper, illustrating appropriate format and style. It presents a proof of finitary Ramsey's theorem (Ramsey, 1930) from infinitary Ramsey's theorem (Ramsey, 1930) and the compactness theorem (Gödel, 1929).

1 Introduction

In 1930, Frank Ramsey published proofs of both finitary and infinitary versions of what is now known as Ramsey's theorem [3]. In the same year, Kurt Gödel published results from his 1929 doctoral thesis, including the compactness theorem for first-order logic [2]. The finitary version of Ramsey's theorem can be proven from the infinitary version with the help of the compactness theorem. In this paper, I will explain that proof.

Let's begin with a classic version of a special case of Ramsey's theorem: Suppose there are six people in a room. Then either there are three people in the room all of whom know each other, or there are three people in the room none of whom know each other.

Here is another version of the same mathematical fact: Draw six points arranged in a hexagon, draw all the line segments connecting pairs of those points, and color each line segment red or blue. Then your drawing will contain either a red triangle or a blue triangle.

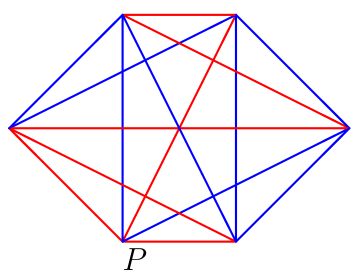


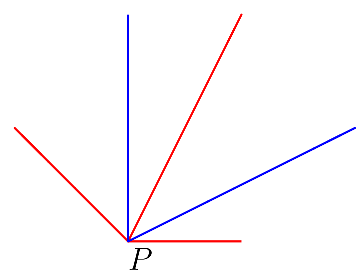
This is a version of the same fact, because we can let the points represent people, the red lines connect people who know each other, and the blue lines connect people who do not know each other.

The mathematical fact is this: If X is any set of size 6 or greater, and the collection of (unordered) pairs from X is partitioned into two pieces (for example, acquainted and unacquainted, or red and blue), then there is a subset $Y \subset X$ of size 3, such that all the two-element subsets of Y are in the same piece.

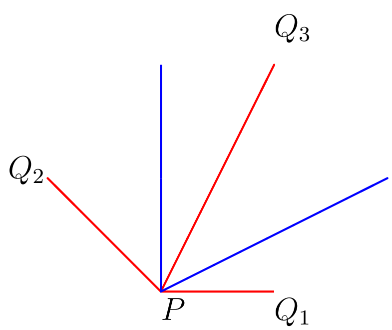
To prove this, let's consider the colored line segment version: X consists of six points, each pair of elements of X determines a line segment connecting those two points, and each line segment is colored red or blue. We will find either a red triangle or a blue triangle.

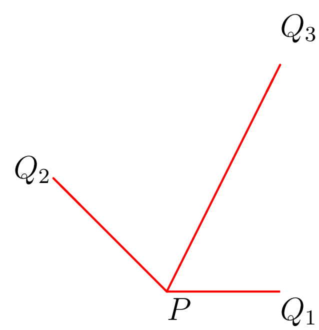
Pick any point P to start with. There are five line segments connecting P to the other five points, so at least three of them must be the same color.



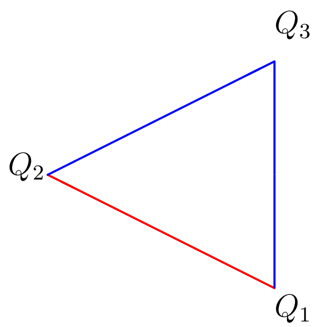


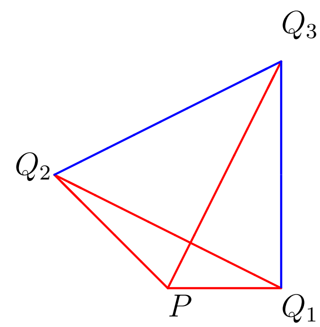
Suppose the line segments connecting P to Q_1, Q_2, and Q_3 are all red.



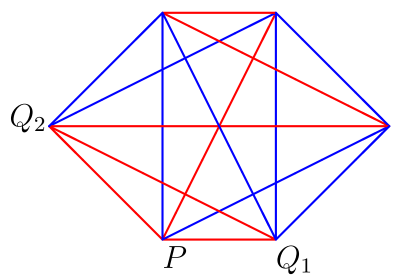


If all the line segments with endpoints Q_i and Q_j are blue, then Q_1, Q_2, and Q_3 are the corners of a blue triangle. If not, say the line segment between Q_1 and Q_2 is red.





Then P, Q_1, and Q_2 are the corners of a red triangle.



The finitary version of Ramsey's theorem is a generalization of the fact we have just proved. In Section 2, I will give some definitions and state both the finitary and the infinitary versions of Ramsey's theorem. In Section 3, I will give a proof of the infinitary version, and in Section 4, I will use the compactness theorem to prove the finitary version from the infinitary version.

2 Definitions

To state the general fact of which we have seen a special case, we need some definitions.

Definition 2.1. Let n and k be positive natural numbers.

If X is any set, $X^{[n]}$ denotes the collection of n-element subsets of X:

$$X^{[n]} = \{Y \subseteq X \mid \text{size}(Y) = n\}.$$

A coloring of $X^{[n]}$ in k colors is a function $f : X^{[n]} \to C$, where C is a k-element set. We call the elements of C colors, and we think of f as assigning a color to each n-element subset of X.

An n-coloring of X is a coloring of $X^{[n]}$. That is, it assigns to each n-element subset of X a color, chosen from some set C of colors.

If f is an n-coloring of X, a subset $Y \subseteq X$ is *homogeneous* or *monochromatic* for f if there is some $i \in C$ such that, for every $s \in Y^{[n]}$, we have $f(s) = i$. We call Y monochromatic because all n-element subsets of Y have the same color i.

In the terms of this definition, the fact we proved in Section 1 is the following: If X is any set of size at least 6, then for every 2-coloring of X in 2 colors, there is a monochromatic subset Y of size 3.

Theorem 2.2 (Finitary Ramsey's Theorem (Ramsey, 1930) [3])**.** *For all positive natural numbers n, k, and a, there is a natural number b such that if X is any set of size at least b, then for every n-coloring of X in k colors, there is a monochromatic subset $Y \subseteq X$ of size a.*

Finitary Ramsey's theorem says that you may arbitrarily choose the size n of the sets that are assigned colors, the number k of colors, and the size a of the (finite) monochromatic set you want. Then there is a (finite) number b large enough so that if your initial set X has size at least b, you are guaranteed to have a monochromatic set of size a.

Infinitary Ramsey's theorem says that you can always get an infinite monochromatic set Y, as long as you start with an infinite set X.

Theorem 2.3 (Infinitary Ramsey's Theorem (Ramsey, 1930) [3])**.** *For all positive natural numbers n and k, for every n-coloring of an infinite set X in k colors, there is an infinite monochromatic subset $Y \subseteq X$.*

It is possible to prove finitary Ramsey's theorem without reference to the infinitary version, but it is a complicated proof. It is easier to prove the infinitary version, and then apply the compactness theorem.

3 Infinitary Ramsey's Theorem

Infinitary Ramsey's theorem says that for all positive natural numbers n and k, for every n-coloring of an infinite set X in k colors, there is an infinite monochromatic subset $Y \subseteq X$. We prove this by induction on n.

For $n = 1$, the theorem states that if we color all the elements of an infinite set X in finitely many colors, then there is an infinite $Y \subseteq X$ all of whose elements are the same color. This is true; if we partition an infinite set into finitely many pieces, one of those pieces must be infinite.

Before giving the inductive step, let's look at the special case where $n = 2$ and $X = \mathbb{N}$. We can represent a two-element subset of $\mathbb{N}$ as an ordered pair (x, y) where we list the smaller element first; geometrically, as a point in the plane above the diagonal line $x = y$. The picture below illustrates a coloring f of $\mathbb{N}^{[2]}$ in two colors, red (R) and blue (B). (The black numbers label the rows and columns.) The theorem says there is an infinite monochromatic subset $H \subseteq \mathbb{N}$.

8	R	R	B	R	B	B	R	R
7	R	B	B	B	R	B	B	
6	B	R	B	B	R	B		
5	B	R	R	B	R			
4	R	B	R	R				
3	R	B	R					
2	R	R						
1	B							
0								
	0	1	2	3	4	5	6	7

For this coloring, $H = \{0, 7, 20, 24, 30, 34, 55, 56, 81, \ldots\}$ is monochromatic in color red. If we restrict our picture to only the rows and columns corresponding to numbers in H, we see the elements of $H^{[2]}$.

81	R	R	R	R	R	R	R	R
56	R	R	R	R	R	R	R	
55	R	R	R	R	R	R		
34	R	R	R	R	R			
30	R	R	R	R				
24	R	R	R					
20	R	R						
7	R							
0								
	0	7	20	24	30	34	55	56

We will construct a monochromatic set H. First, we construct a preliminary set $Y = \{x_0, x_1, \ldots, x_j \ldots\}$, and a function $c : Y \to \{R, B\}$.

Begin by setting $x_0 = 0$. The points in the 0-column represent sets $\{0, m\}$ with $m > 0$. Since there are infinitely many points in this column, either infinitely many of them are red or infinitely many of them are blue. Say infinitely many are red.

Let X_0 be the set of numbers that pair with 0 to make a red set:

$$X_0 = \{m \in \mathbb{N} - \{0\} \mid f(\{0, m\}) = R\} = \{2, 3, 4, 7, \ldots\}.$$

Geometrically, m is in X_0 just in case R appears in the m^{th} place of the 0-column.

In the following picture, the numbers in rows and columns corresponding to elements of $\{0\} \cup X_0$ have been boxed. These are the zero row and the rows in which an R appears in the 0-column (rows 0,2,3,4,7,...), and the corresponding columns (columns 0,2,3,4,7,...) .

$\boxed{8}$	$\boxed{R}$	R	$\boxed{B}$	$\boxed{R}$	$\boxed{B}$	B	R	$\boxed{R}$
$\boxed{7}$	$\boxed{R}$	B	$\boxed{B}$	$\boxed{B}$	$\boxed{R}$	B	B	
6	B	R	B	B	R	B		
5	B	R	R	B	R			
$\boxed{4}$	$\boxed{R}$	B	$\boxed{R}$	$\boxed{R}$				
$\boxed{3}$	$\boxed{R}$	B	$\boxed{R}$					
$\boxed{2}$	$\boxed{R}$	R						
1	B							
$\boxed{0}$								
	$\boxed{0}$	1	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	5	6	$\boxed{7}$

In the next picture, only the rows and columns corresponding to elements of $\{0\} \cup X_0$ (the boxed items) have been retained. Now the 0-column is entirely red.

14	R	B	B	B	B	R	R	R
11	R	B	R	R	R	B	R	
10	R	R	B	B	R	B		
8	R	B	R	B	R			
7	R	B	B	R				
4	R	R	R					
3	R	R						
2	R							
$\boxed{0}$								
	$\boxed{0}$	2	3	4	7	8	10	11

Because the 0-column is red, we assign the color red to 0; set $c(0) = R$.

Now, we choose our next element of Y. Let x_1 be the smallest element of X_0; that is, set $x_1 = 2$. The points in the 2-column represent sets $\{2, m\}$ with $m \in X_0$ and $m > 2$. Since there are infinitely many such points, either infinitely many of them are red or infinitely many of them are blue. Say infinitely many are blue.

Let X_1 be the set of numbers in X_0 that pair with x_1 (that is, with 2) to make a blue set:

$$X_1 = \{m \in \mathbb{X}_0 - \{x_1\} \mid f(\{x_1, m\}) = B\} = \{7, 8, 11, \dots\}.$$

Since x_1 pairs with elements of X_1 to make blue sets, we assign it color blue; set $c(2) = B$.

$\boxed{14}$	$\boxed{R}$	$\boxed{B}$	B	B	$\boxed{B}$	$\boxed{R}$	R	$\boxed{R}$
$\boxed{11}$	$\boxed{R}$	$\boxed{B}$	R	R	$\boxed{R}$	$\boxed{B}$	R	
10	R	R	B	B	R	B		
$\boxed{8}$	$\boxed{R}$	$\boxed{B}$	R	B	$\boxed{R}$			
$\boxed{7}$	$\boxed{R}$	$\boxed{B}$	B	R				
4	R	R	R					
3	R	R						
$\boxed{2}$	$\boxed{R}$							
$\boxed{0}$								
	$\boxed{0}$	$\boxed{2}$	3	4	$\boxed{7}$	$\boxed{8}$	10	$\boxed{11}$

23	R	B	R	R	B	B	R	R
20	R	B	R	B	B	B	B	
18	R	B	R	R	R	R		
14	R	B	B	R	R			
11	R	B	R	B				
8	R	B	R					
7	R	B						
$\boxed{2}$	R							
$\boxed{0}$								
	$\boxed{0}$	$\boxed{2}$	7	8	11	14	18	20

Now we continue in the same way to choose the remaining elements of Y:

$$x_j = min(X_{j-1});$$

$$X_j = \begin{cases} \{m \in X_{j-1} - \{x_j\} \mid f(\{x_j, m\}) = R\} & \text{if this is infinite;} \\ \{m \in X_{j-1} - \{x_j\} \mid f(\{x_j, m\}) = B\} & \text{otherwise;} \end{cases}$$

$$c(x_j) = \begin{cases} R \\ B \end{cases} \quad \text{respectively.}$$

Let $Y = \{x_0, x_1, x_2, \ldots, x_j, \ldots\}$. If we restrict to the rows and columns corresponding to elements of Y, each column has only one color. The color of the j-column is $c(j)$.

30	R	B	R	B	B	R	B	R
24	R	B	R	B	B	R	B	
23	R	B	R	B	B	R		
20	R	B	R	B	B			
11	R	B	R	B				
8	R	B	R					
7	R	B						
2	R							
0								
	0	2	7	8	11	20	23	24

Either there are infinitely many red columns or there are infinitely many blue columns. Say there are infinitely many red columns. Then

$$H = \{m \in Y \mid \text{ the } m\text{-column is red }\} = \{m \in Y \mid c(m) = R\}$$

is infinite.

The next picture on the left indicates rows and columns corresponding to elements of H, and the one on the right retains only those rows and columns.

$\boxed{30}$	$\boxed{R}$	B	$\boxed{R}$	B	B	$\boxed{R}$	B	$\boxed{R}$
$\boxed{24}$	$\boxed{R}$	B	$\boxed{R}$	B	B	$\boxed{R}$	B	
23	R	B	R	B	B	R		
$\boxed{20}$	$\boxed{R}$	B	$\boxed{R}$	B	B			
11	R	B	R	B				
8	R	B	R					
$\boxed{7}$	$\boxed{R}$	B						
2	R							
$\boxed{0}$								
	$\boxed{0}$	2	$\boxed{7}$	8	11	$\boxed{20}$	23	$\boxed{24}$

81	R	R	R	R	R	R	R	R
56	R	R	R	R	R	R	R	
55	R	R	R	R	R	R		
34	R	R	R	R	R			
30	R	R	R	R				
24	R	R	R					
20	R	R						
7	R							
0								
	0	7	20	24	30	34	55	56

H is an infinite monochromatic set in color R.

This argument illustrates the inductive step of the proof. Every time we say "there must be infinitely many red points or infinitely many blue points," we are using the fact that for every 1-coloring of an infinite set in 2 colors there is an infinite monochromatic set. That is, we are using the inductive hypothesis for 1-colorings to prove the theorem for 2-colorings.

For the inductive step of the proof of infinitary Ramsey's theorem, assume as inductive hypothesis that for every n-coloring of an infinite set in k colors there is an infinite monochromatic set, and let $f : X^{[n+1]} \to C$ be an $(n+1)$-coloring of an infinite set X in k colors. We must show there is an infinite monochromatic set for f.

As before, we will first construct a preliminary set $Y = \{x_0, x_1, \ldots, x_j, \ldots\}$ and a function $c : Y \to C$.

Let $X_{-1} = X$.

Suppose X_{j-1} has been defined. Let x_j be any element of X_{j-1}. Let $Y_{j-1} = X_{j-1} - \{x_j\}$. We define an n-coloring of Y_{j-1}, a function $f_j : (Y_{j-1})^{[n]} \to C$, as follows:

$$f_j(\{y_1, \ldots, y_n\}) = f(\{x_j, y_1, \ldots, y_n\}).$$

Then, by the inductive hypothesis, we let $X_j \subseteq Y_{j-1}$ be an infinite set monochromatic for f_j in some color $i \in C$, and we set $c(x_j) = i$. That is, X_j is an infinite subset of X_{j-1}, all of whose n-element subsets join with x_j to form $(n+1)$-element sets of color $c(x_j)$.

Let

$$Y = \{x_j \mid j \in \mathbb{N}\}.$$

By construction, we have $X = X_{-1} \supseteq X_0 \supseteq X_1 \supseteq \cdots$, and for $j < \ell$ we have $x_\ell \in X_{\ell-1} \subseteq X_j$.

Suppose $\{x_j, x_{\ell(1)}, \ldots, x_{\ell(n)}\}$ is an $(n+1)$-element subset of Y, ordered so $j < \ell(1) \cdots < \ell(n)$. Because $\{x_{\ell(1)}, \ldots, x_{\ell(n)}\} \subset X_j$, by our choice of X_j, we have $f(\{x_j, x_{\ell(1)}, \ldots, x_{\ell(n)}\}) = c(x_j)$. That is, if F is any $(n+1)$-element subset of Y and j is the smallest index such that $x_j \in F$, then $f(F) = c(x_j)$.

Because Y is infinite and C is finite, we can choose a color $i \in C$ such that the set $\{x_j \in Y \mid c(x_j) = i\}$ is infinite. Choose such a color, and let

$$H = \{x_j \in Y \mid c(x_j) = i\}.$$

Because $H \subseteq Y$, if F is any $(n+1)$-element subset of H and j is the smallest index such that $x_j \in F$, then $f(F) = c(x_j)$. But since $x_j \in H$, we have $c(x_j) = i$, so $f(F) = i$. That is, H is an infinite monochromatic set in color i.

This completes the proof of infinitary Ramsey's theorem.

You might have noticed that this proof requires making infinitely many arbitrary choices of elements $x_k \in X_{k-1}$ and of monochromatic sets X_k, and therefore it seems to require the axiom of choice. It does. The proof does not actually use the full axiom of choice, but a weak version, called the axiom of dependent choice. However, if the axiom of dependent choice is false, it is possible for infinitary Ramsey's theorem to be false as well.

4 Finitary Ramsey's Theorem

Recall the statement of finitary Ramsey's theorem:

> For all positive natural numbers n, k, and a, there is a natural number b such that if X is any set of size at least b, then for every n-coloring of X in k colors, there is a monochromatic subset $Y \subseteq X$ of size a.

We wish to prove this from infinitary Ramsey's theorem and the compactness theorem for first-order logic.

Before stating the compactness theorem, I will need a couple of definitions. (These definitions, and other facts about first-order logic, can be found in Enderton's textbook [1].) A structure is a *model* for a set of sentences Γ, or *satisifies* Γ, if every sentence in Γ is true in that structure. For example, a model of the group axioms is a group. A set of sentences Γ is *satisfiable* if it has a model, and *finitely satisfiable* if every finite subset has a model.

> Compactness Theorem (Gödel, 1929) [2]: Every finitely satisfiable set of sentences (in a countable first-order formal language) is satisfiable.

We will use this to prove finitary Ramsey's theorem from infinitary Ramsey's theorem.

Suppose that finitary Ramsey's theorem is false. Then there are positive natural numbers n, k, and a, such that

> (*) For all natural numbers b, there is a set X of size at least b and there is an n-coloring of X in k colors with no monochromatic subset $Y \subseteq X$ of size a.

We will use this to produce a certain finitely satisfiable set of sentences Γ. From the compactness theorem, we will conclude that Γ is satisfiable, and from infinitary Ramsey's theorem, we will conclude that Γ is not satisfiable. This contradiction proves finitary Ramsey's theorem.

First we define the set of sentences Γ. Let $\mathcal{L}$ be the first-order language with equality and n-place predicate symbols $P_1, \ldots, P_k$. We intend $P_i x_1 \ldots x_n$ to mean "the set $\{x_1, \ldots, x_n\}$ has color i."

Let ψ be a sentence saying that the P_i define a coloring of n-element sets. I will leave the technical details to the end of the proof. Roughly, ψ says that the truth of $P_i x_1 \dots x_n$ does not depend on the order in which the x_j are presented (the P_i define a coloring of n-element sets, not of ordered n-tuples), $P_i x_1 \dots x_n$ does not hold if any two of the x_j are equal (only sets of size exactly n are colored), and if no two of the x_j are equal then exactly one of $P_1 x_1 \dots x_n$, $P_2 x_1 \dots x_n$, $\dots$, $P_k x_1 \dots x_n$ holds (each n-element set is assigned a unique color).

For each natural number b, let σ_b be a sentence asserting the existence of at least b-many elements. Let γ be a sentence that says no a-element set is monochromatic. Again, I leave the technical details to the end of the proof.

Let

$$\Gamma_b = \{\psi, \gamma, \sigma_0, \sigma_1, \dots, \sigma_b\}.$$

A model of Γ_b is a set of size at least b with an n-coloring in k colors, and with no monochromatic subset of size a. Our assumption

> (*) for all natural numbers b, there is a set X of size at least b and there is an n-coloring of X in k colors with no monochromatic subset $Y \subseteq X$ of size a

asserts that every Γ_b has a model; every Γ_b is satisfiable.

Now let

$$\Gamma = \bigcup_{b \in \mathbb{N}} \Sigma_b = \{\psi, \gamma, \sigma_0, \sigma_1, \dots, \sigma_b, \dots\}.$$

A model of Γ is an infinite set with an n-coloring in k colors, and with no monochromatic subset of size a. Because every finite subset of Γ is contained in some Γ_b, every finite subset has a model; Γ is finitely satisfiable. By the compactness theorem, therefore, Γ is satisfiable.

Suppose, then, that the infinite set X with coloring f is a model of Γ. That is, there is no monochromatic a-element subset of X. By infinitary Ramsey's theorem, however, X must have an infinite monochromatic subset H. Every a-element subset of H is monochromatic. This is a contradiction.

To complete the proof of finitary Ramsey's theorem, it remains only to give the details of the definitions of ψ, γ, and σ_b.

The symbol $\wedge$ denotes the conjunction of two formulas; $(\alpha \wedge \beta)$ means "α and β." The symbol $\bigwedge$ denotes the conjunction of a finite collection of formulas; $\bigwedge_{1 \le i < j \le 3} \alpha_{i,j}$ means "$\alpha_{1,2}$ and $\alpha_{1,3}$ and $\alpha_{2,3}$." Similarly, $\vee$ and $\bigvee$ denote disjunction (inclusive or).

Definition 4.1 (definition of ψ)**.** For $1 \le i \le k$, let τ_i be a sentence that says $P_i x_1 \dots x_n$ holds only if the x_j are all distinct (sets of size less than n are not colored), and that the truth or falsity of $P_i x_1 \dots x_n$ does not depend on the ordering of the x_j (whether $\{x_1, \dots, x_n\}$ has color i does not depend on the order in which the elements are listed). For $n = 2$, the sentence τ_i is

$$\forall x \forall y \left((P_i xy \to x \neq y) \wedge (P_i xy \leftrightarrow P_i yx) \right).$$

In general, the sentence τ_i is

$$\forall x_1 \dots \forall x_n \left(\left(P_i x_1 \dots x_n \to \bigwedge_{1 \le \ell < j \le n} x_\ell \ne x_j \right) \wedge \bigwedge_{g \in S_n \;\; g \ne id} (P_i x_1 \dots x_n \leftrightarrow P_i x_{g(1)} \dots x_{g(n)}) \right),$$

where S_n is the set of all permutations (reorderings) of $\{1, 2, \dots, n\}$.

For $1 \le i, j \le k$, $i \ne j$, let $\rho_{i,j}$ be a sentence that says if P_i holds, then P_j does not (no set can be colored both color i and color j). The sentence $\rho_{i,j}$ is

$$\forall x_1 \dots \forall x_n (P_i x_1 \dots x_n \to \neg P_j x_1 \dots x_n).$$

Let π be a sentence that says at least one P_i holds for every n-element set (every n-element set is colored with some color). For $n = 2$ and $k = 3$, the sentence π is

$$\forall x \forall y \, (x \ne y \to P_1 xy \vee P_2 xy \vee P_3 xy) \, .$$

In general, the sentence π is

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{1 \le \ell < j \le n} x_\ell \ne x_j \to \bigvee_{1 \le i \le k} P_i x_1 \dots x_n \right).$$

Let ψ be the conjunction of the τ_i, the $\rho_{i,j}$, and π. Then ψ says that the P_i define a coloring of n-element sets. In any model of ψ, the coloring is determined by defining the color of $\{a_1, \dots, a_n\}$ to be the unique i such that the model satisfies the formula $P_i v_1, \dots, v_n$ when the variables $v_1, \dots, v_n$ are assigned to the elements $a_1, \dots, a_n$.

Definition 4.2 (definition of γ)**.** For $1 \le i \le k$, let $\varphi_i(x_1, \dots, x_a)$ say that $\{x_1, \dots, x_a\}$ has a subset for which P_i holds; that is, the color i appears in this set. For $n = 2$ and $a = 4$, the formula $\varphi_i(x_1, x_2, x_3, x_4)$ is

$$P_i x_1 x_2 \vee P_i x_1 x_3 \vee P_i x_1 x_4 \vee P_i x_2 x_3 \vee P_i x_2 x_4 \vee P_i x_3 x_4.$$

In general, the formula $\varphi_i(x_1, \dots, x_1)$ is

$$\bigvee_{1 \le j(1) < j(2) < \dots < j(n) \le a} P_i x_{j(1)} x_{j(2)} \dots x_{j(n)}.$$

Now let γ be a sentence that says there is no monochromatic set of size a; that is, in any set of size a, at least two distinct colors appear. The sentence γ is

$$\forall x_1 \dots \forall x_a \left(\bigwedge_{1 \le \ell < j \le a} x_\ell \ne x_j \to \bigvee_{1 \le h < i \le k} (\varphi_h(x_1, \dots, x_a) \wedge \varphi_i(x_1, \dots, x_a)) \right).$$

A model of $\{\psi, \gamma\}$ is a set with an n-coloring in k colors for which there is no monochromatic set of size a.

Definition 4.3 (definition of σ_b)**.** For each natural number $b \geq 2$, let σ_b be a sentence that says there are at least b elements: σ_0 is $\forall x(x = x)$, σ_1 is $\exists x(x = x)$, and for $b \geq 2$, σ_b is

$$\exists x_1 \ldots \exists x_m \bigwedge_{1 \leq i < j \leq m} (x_i \neq x_j).$$

Any structure satisfying σ_b has at least b-many elements, and any structure satisfying all the σ_b is infinite.

This completes the proof of finitary Ramsey's theorem.

5 Final Comments

While this proof is easier than the direct proof of finitary Ramsey's theorem, the direct proof has some advantages as well. Most notably, from the direct proof, you can begin to answer the question of how big b must be, for a given n, k, and a. (The answer is, very big indeed.)

Arguments from the compactness theorem, like this one, are not uncommon in some areas of combinatorics; in particular, in generalized Ramsey theory.

References

[1] Enderton, Herbert B. *A Mathematical Introduction to Logic*, second edition. Harcourt/Academic Press, 2001.

[2] Gödel, Kurt. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatshefte für Mathematik* 37 (1): 349–360, 1930.

[3] Ramsey. F.P. On a problem of formal logic. *Proc. London Math. Soc. Ser. 2* 30:264–286, 1930.

Notes for Math 69 authors:

You need not go as far as I did in explaining your notation. That is, you may assume your reader knows about first-order languages, and understands the basic notation in our textbook. Think of your reader as another student in the class, who may not understand the material quite as well as you do,[1] and who of course does not know anything about the specific problem you are writing about. I intended this paper for a possibly wider audience.

Pictures and diagrams can be helpful sometimes. However, unless you have time on your hands and are up for a project, you don't need to learn how to draw them electronically. Hand-drawn pictures, provided they are clear and neat, are perfectly acceptable.

The authorial "we" and the authorial "I" are both acceptable, although you should be consistent. Most people will tell you to use one or the other. In this paper, I used both, but (I hope) consistently; I used "I" when talking about authorial choices ("I will leave these details for later,") and "we" in proofs, implicitly assuming the reader is thinking through the proof along with me ("We have obtained a contradiction.") This is my own peculiarity, which you probably shouldn't emulate unless you feel very comfortable with it.

There is some debate about whether one should write "Ramsey's theorem" or "Ramsey's Theorem," but the first seems more standard in modern usage.

The citation style I used is standard for mathematics papers. You may use this, or any footnote or endnote style you are used to, as long as you are consistent and include a list of references. You may not use a style that places citations parenthetically in the text; mathematics is hard enough to read without cluttering it up with things that could very well go elsewhere.

Most of what you need to know about format and style is addressed in one of the papers linked to from the course web page. It is described as a paper about how to write a mathematics paper, and you can find the link on the General Information page under Exams.

Excellent mathematical writing style embodies several characteristics, of which the three most important are clarity, clarity, and clarity. It is important to use words precisely and correctly. Generally, simple declarative sentences and consistent word use are preferable to variation in sentence structure and vocabulary. The same is true of most technical writing; the deeper and more complex the ideas, the more simple the writing should be. My favorite quotation about this comes from the web page "Guidelines for Writing a Phiilosophy Paper" by NYU philosophy professor James Pryor:[2]

> If your paper sounds as if it were written for a third-grade audience, then you've probably achieved the right sort of clarity.

[1] Even if you think you are the worst student in the class, assume your reader does not understand things as well as you do. It will enhance the clarity of your prose.

[2] http://www.jimpryor.net/teaching/guidelines/writing.html