Math 69
Winter 2009
Wednesday, January 7

Deductions, Soundness and Completeness

If Σ is a set of formulas and α is a formula, we define a *deduction* of α from Σ to be a finite sequence of formulas

$$\begin{array}{l}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{array}$$

whose last element α_n is α, such that for each $i \leq n$, one of the following conditions holds:

1. The formula α_i is a tautology (a *logical axiom.*)
2. The formula α_i is a member of Σ (a *hypothesis.*)
3. There are j and k less than i such that

   $\alpha_k = (\alpha_j \to \alpha_i)$.

   (The formula α_i is derived from α_j and α_k by *modus ponens.*)

If there is a deduction of α from Σ, we write $\Sigma \vdash \alpha$. For example, $\{(A \wedge B)\} \vdash A$, as shown by the following deduction:

$$\begin{array}{l}(A \wedge B)\\ ((A \wedge B) \to A)\\ A\end{array}$$

You should be able to explain why each line is legitimate according to the definition of deduction.

Soundness Theorem: If $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$.

Prove the Soundness Theorem. Suggestion: Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction of α from Σ, and prove by (strong) induction on i that $\Sigma \models \alpha_i$ for each $i \leq n$.

You might or might not want to separately prove as a lemma that, for any set of wffs Σ and any wffs β and γ, if $\Sigma \models \beta$ and $\Sigma \models (\beta \to \gamma)$, then $\Sigma \models \gamma$.

We stated in class last time that, according to our convention, "Every truth valuation satisfying Σ also satisfies α" is true in case there are no truth valuations that satisfy Σ. Therefore,

$$\{A, (\neg A)\} \models B.$$

Show that also

$$\{A, (\neg A)\} \vdash B.$$

Completeness Theorem: If $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$.

Prove the Completeness Theorem. Suggestion: Use the Compactness Theorem to prove that if $\Sigma \models \alpha$ then there is a finite $\Gamma \subset \Sigma$ such that $\Gamma \models \alpha$. (Recall that you proved Wednesday that $\Sigma \models \alpha$ iff $\Sigma \cup \{(\neg\alpha)\}$ is not satisfiable.)

You might want to prove separately as a lemma that

$$(\alpha_n \rightarrow (\alpha_{n-1} \rightarrow (\cdots \alpha_1 \rightarrow \beta) \cdots))$$

is a tautology if and only if

$$\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1\} \models \beta.$$

Preview of a homework problem from the textbook: In 1977 it was proved that every planar map can be colored with four colors. Of course, the definition of "map" requires that there be only finitely many countries. But extending the concept, suppose we have an infinite (but countable) planar map with countries $C_1, C_2, C_3, \ldots$. Prove that this infinite planar map can still be colored with four colors. (Suggestion: Partition the sentence symbols into four parts. One sentence symbol, for example, can be used to translate, "Country C_7 is colored red." Form a set Σ_1 of wffs that say, for example, C_7 is exactly one of the colors. Form another set Σ_2 of wffs that say, for each pair of adjacent countries, that they are not the same color. Apply compactness to $\Sigma_1 \cup \Sigma_2$.)

Note: The intention here is that you are given a *specific* infinite map, and you use this map to produce your sets of wffs. That is, for each infinite countable planar map $\mathcal{M}$, there is a set $\Sigma_{\mathcal{M}}$ of wffs, such that applying compactness to $\Sigma_{\mathcal{M}}$ demonstrates that $\mathcal{M}$ can be colored with four colors.