Math 69
Winter 2009
Wednesday, January 7

Truth Assignments and Tautological Implication

If v is a truth assignment on the set of sentence symbols (v is a function that assigns each sentence symbol to T or F), we extend v to a truth assignment $\overline{v}$ on all formulas by recursion on formulas:

$$\overline{v}(A_n) = v(A_n)$$

$$\overline{v}((\neg\alpha)) = \begin{cases} T & \overline{v}(\alpha) = F \\ F & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \wedge \beta)) = \begin{cases} T & \overline{v}(\alpha) = \overline{v}(\beta) = T \\ F & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \vee \beta)) = \begin{cases} F & \overline{v}(\alpha) = \overline{v}(\beta) = F \\ T & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \rightarrow \beta)) = \begin{cases} F & \overline{v}(\alpha) = T \;\&\; \overline{v}(\beta) = F \\ T & \text{otherwise.} \end{cases}$$

$$\overline{v}((\alpha \leftrightarrow \beta)) = \begin{cases} T & \overline{v}(\alpha) = \overline{v}(\beta) \\ F & \text{otherwise.} \end{cases}$$

Another way to phrase this is using Boolean functions as discussed in Monday's handout. That is, we can define

$$Val_{\neg}(X) = \begin{cases} T & X = F \\ F & X = T \end{cases}$$

$$Val_{\wedge}(X, Y) = \begin{cases} T & X = Y = T \\ F & \text{otherwise.} \end{cases}$$

$$Val_{\vee}(X, Y) = \begin{cases} F & X = Y = F \\ T & \text{otherwise.} \end{cases}$$

$$Val_{\rightarrow}(X,Y) = \begin{cases} F & X = T \;\&\; Y = F \\ T & \text{otherwise.} \end{cases}$$

$$Val_{\leftrightarrow}(X,Y) = \begin{cases} T & X = Y \\ F & \text{otherwise.} \end{cases}$$

Then we can define $\overline{v}$ by

$$\overline{v}(A_n) = v(A_n)$$

$$\overline{v}((\neg\alpha)) = Val_{\neg}(\overline{v}(\alpha)),$$

and for any binary connective $*$,

$$\overline{v}((\alpha * \beta)) = Val_{*}(\overline{v}(\alpha), \overline{v}(\beta)).$$

This notation might simplify the following task.

Prove this **Propositon:** For every two truth assignments v and w that agree with each other on every sentence symbol that occurs in α, we have $\overline{v}(\alpha) = \overline{w}(\alpha)$.

Show that the following are tautologically equivalent:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \rightarrow \beta$$

$$\alpha_n \rightarrow (\alpha_{n-1} \rightarrow (\cdots(\alpha_1 \rightarrow \beta)\cdots))$$

(Of course, neither of the above is actually a formula. We will eliminate parentheses when that can be done unambiguously; the textbook gives rules for eliminating parentheses at the end of section 1.3. Officially, the first formula given above is an abbreviation for the actual wff

$$((\alpha_1 \wedge (\alpha_2 \wedge (\cdots \wedge \alpha_n)\cdots)) \rightarrow \beta).$$

You should avoid like the plague omitting parentheses in formulas involving $\rightarrow$ and $\leftrightarrow$. Officially, for example, $A \leftrightarrow B \leftrightarrow C$ is an abbreviation for $(A \leftrightarrow (B \leftrightarrow C))$, and does NOT mean that A, B, and C have the same truth value. The textbook may abbreviate the second formula above as

$$\alpha_n \rightarrow \alpha_{n-1} \rightarrow \cdots \rightarrow \alpha_1 \rightarrow \beta.).$$

Show the following:

$\Sigma \models \alpha$ if and only if $\Sigma \cup \{\neg\alpha\}$ is not satisfiable.

If Σ is satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is satisfiable.

A set of formulas Σ is said to be *finitely satisfiable* if every finite subset of Σ is satisfiable. We are about to prove the Compactness Theorem: If Σ is finitely satisfiable, then Σ is satisfiable. Prove the following proposition, which we will use as a lemma:

Proposition: If Σ is finitely satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.

Here is an outline of the proof of the Compactness Theorem:

Suppose that Σ is finitely satisfiable. Define, by induction on n,

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\} & \text{if this is finitely satisfiable;} \\ \Sigma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Show that each Σ_n is finitely satisfiable.

Now let $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$. Show that Σ^* is finitely satisfiable.

Note that $\Sigma \subseteq \Sigma^*$, and that for each n, either A_n or $\neg A_n$ is in Σ^* (but not both). Define a truth valuation v by

$$v(A_n) = \begin{cases} T & A_n \in \Sigma^* \\ F & \neg A_n \in \Sigma^* \end{cases}$$

Show that v satisfies Σ^*, and therefore Σ, as follows:

Suppose not. Let $\alpha \in \Sigma^*$ with $\overline{v}(\alpha) = F$. For each sentence symbol A_n, define

$$\beta_n = \begin{cases} A_n & A_n \in \Sigma^* \\ \neg A_n & \neg A_n \in \Sigma^* \end{cases}$$

Let Γ be a finite subset of Σ^* containing α and β_n for every sentence symbol A_n that occurs in α. Because Σ^* is finitely satisfiable, there is a truth assignment w satisfying Γ. Deduce a contradiction.