Math 68 - Algebraic Combinatorics

Lattices

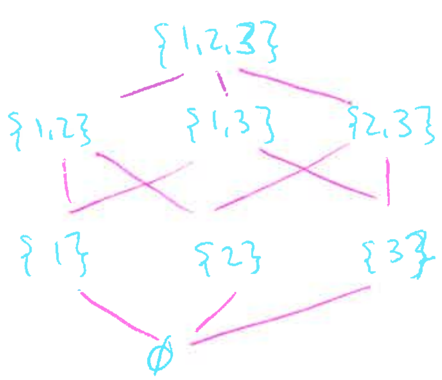
Nadia Lafrenière
10/02/2019

Definitions

Two elements s and t in a poset have a

- least upper bound u, called the join, if $u \geq s$, $u \geq t$, and if $v > s$ and $v > t$, then $v \geq u$. We denote it $u = s \vee t$ ("s join t").
- greatest lower bound w, called the meet, if $w \leq s$, $w \leq t$, and if $v < s$ and $v < t$, then $v \leq w$: $w = s \wedge t$ ("s meet t")

To remember the symbols and names, I think of the poset of subsets ordered with inclusion:



- join is union, $\vee \longleftrightarrow \cup$
- meet is intersection, $\wedge \longleftrightarrow \cap$

Properties of meet and join

(i) $x \wedge x = x \vee x = x$ — idempotent

(ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ — commutativity

(iii) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ — associativity

(iv) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ — absorption.

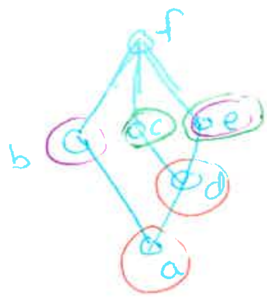
for any objects x and y in a poset

Definition

A lattice is a poset in which each pair has a meet and a join.

Example

Consider the lattice

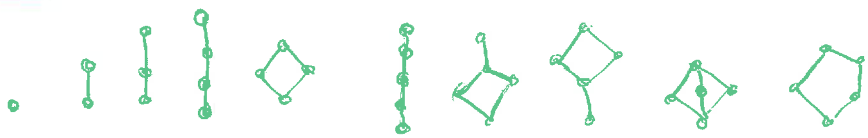


$e \wedge b = a$, $e \vee b = f$

$e \wedge c = d$, $e \vee c = f$

$a \wedge d = a$, $a \vee d = d$.

Example: All the lattices with at most 5 elements



The number of lattices with n elements is the sequence

$$1, 1, 1, 2, 5, 15, \ldots$$

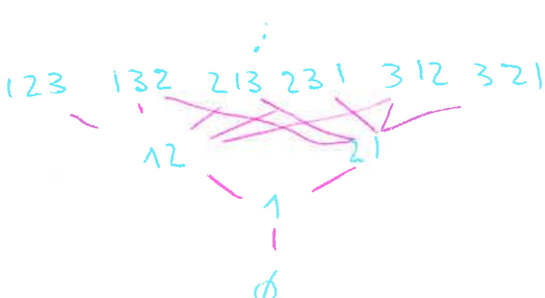
This sequence is known up to $n = 20$ (with approx. 2.3×10^{13} lattices)

Example

All the natural posets from last lecture are lattices:

- chain of order n: $[n]$
- subsets of $[n]$, or boolean lattice
- divisors lattice
- refinement for set partitions
- Tamari lattice
- dominance order

However, the permutation patterns poset arises naturally and is not a lattice



A pattern is a subpermutation of a longer permutation. For example, 123 is an increasing sequence of length 3.

The pattern poset is not a lattice, since 12 and 21 do not have a (unique) join.

Note that, by definition, lattices must be connected.

Proposition

(i) If a lattice is finite, it has a $\hat{0}$ and a $\hat{1}$.

(ii) The dual of a lattice is a lattice

(iii) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$, for all x, y, z in a lattice.

Proof

(i) and (ii) are straightforward.

The two inequalities in (iii) are similar, so we only prove the first one.

First note that $y \vee z \geq y$, and that means that

$$x \wedge (y \vee z) \geq x \wedge y.$$

For the same reason, $x \wedge (y \vee z) \geq x \wedge z$.

The right-hand side of the inequality is the least upper bound of $x \wedge z$ and $x \wedge y$, so it cannot be bigger than something already bigger than $x \wedge y$ and $x \wedge z$, namely $x \wedge (y \vee z)$.

Note

The inequalities in (iii) can be strict equalities in a specific case, given by the following theorem.

Theorem

The following conditions are equivalent, on a lattice L with $x, y, z \in L$.

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

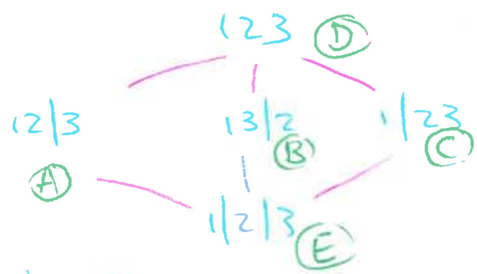
(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(iii) The lattice L is distributive.

(iii) is a definition but the reader should check that (i) $\Leftrightarrow$ (ii)
The proof can be found in [AOC], proposition 5.3.3

Examples

- The chain lattice ($[n]$), the boolean lattice, the divisors lattice are distributive lattice.
- The poset of set partitions of 3 is not distributive



$$A \wedge (B \vee C) = A$$

$$(A \wedge B) \vee (A \wedge C) = E \vee E = E$$

Definition

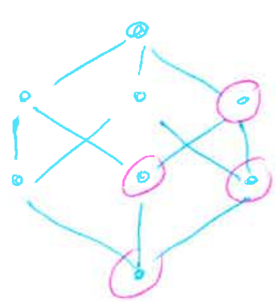
Let $(P, \leq)$ be a poset, and $I \subseteq P$. I is an order ideal if, for any $x \in I$

$$y \leq x \Rightarrow y \in I.$$

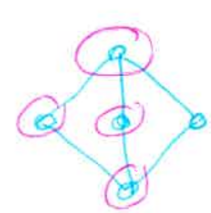
This is very easy to see on a Hasse diagram, since it corresponds to taking all the points "below" a generating set.

Examples

Order ideal



Not an order ideal



Denote $J(P)$ the set of all order ideals of P.

Proposition

If P is any poset, $(J(P), \subseteq)$ is a distributive lattice.

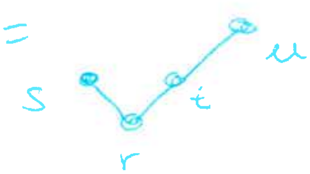
Sketch of proof

To prove that this is a lattice, we must find the meet and the join. For I and $J \in J(P)$, those are respectively the intersection and the union. It is sufficient to show that since union and intersection of sets satisfy distributive laws.

What we will see soon is that all distributive lattices can be recovered in this way.

Example

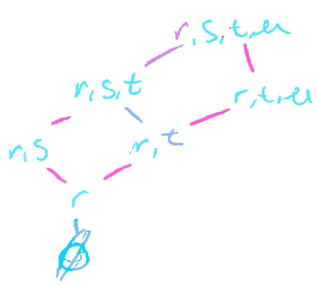
$P =$



$J(P)$

generator(s)	ideal
$\emptyset$	$\emptyset$
r	r
s	r, s
t	r, t
u	r, t, u
s, t	s, r, t
s, u	s, r, t, u

$J(P) =$

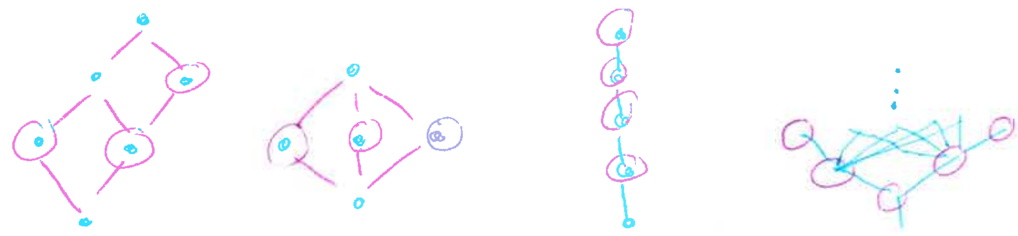


Definition

Let L be a lattice and $x \in L$. x is join-irreducible if $x \neq \hat{0}$ and it cannot be written as $x = y \vee z$, $y, z < x$.
Equivalently, x covers exactly one element.
Denote $Irr(L)$ the set of join irreducible.

Example

The join irreducible elements are circled below.



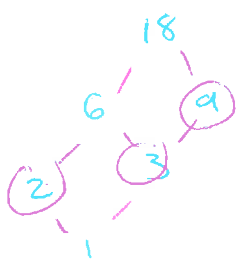
D_{18} $\quad$ Π_3 $\quad$ C_5 $\quad$ patterns

Theorem (Birkhoff; Fundamental Theorem of Finite Distributive Lattices, FTFDL)

If L is a finite distributive lattice, then $L \cong J(P)$, where $P = Irr(L)$. (The order on $J(P)$ is inclusion).

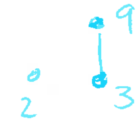
Example

Let L be the posets of divisors of 18.



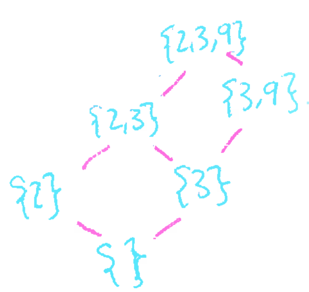
The join irreducible elements of it are 2, 3 and 9.

They form the induced poset



The order ideal of the latter are $\{\}, \{2\}, \{3\}, \{2,3\}, \{3,9\}, \{2,3,9\}$.

Thus, $J(P)$ has the following diagram:



Do you notice anything special about the labels?

Reference: [AOC] Bruce E. Sagan. Combinatorics, the art of counting. Section 5.3