Math 68- Catalan numbers

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- What are Catalan numbers?
    - A sequence of numbers that appear in <u>many</u> places in combinatorics.
    - The longest entry in OEIS (sequence A000108)
    - The first terms are

      1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...

      and it grows fast.
    - Richard Stanley listed 214 objects that are counted by Catalan numbers. They all appear in its book <u>Catalan numbers</u> (2015).

- They count triangulations of convex polygons:
    - A <u>triangulation</u> of a convex polygon with $n+2$ sides is a set of $n-1$ diagonals of the polygons that do not cross (in their interiors). They split the $(n+2)$-gon into n triangles.
    - There are C_n triangulations of a $(n+2)$-gon.

triangles 

quadrilaterals 

pentagons 

hexagons  + 5 rotations

 + 2 rotations  + 2 rotations

 + 1 rotation

Total = 14 triangulations

- Any recursive formula?

  - The number of triangulations of an $(n+2)$-gon can be deduced in the following way.

1. Draw it, with a triangulation.
2. Fix an edge and remove it. Close the polygon by drawing two edges inside, where the diagonals are.
3. The number of triangulations is the product of the numbers of triangulations of the two polygons with fewer sides.

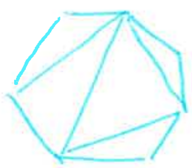
Note that the sum of sides of the smaller polygons will always be $n+3$, but the number in each is fixed by the initial triangulation. So, to count the total number of triangulations, we have to sum over the possible numbers of sides for the first (counterclockwise) smaller polygon.

We get

$$\underbrace{C_n}_{\#\text{ triangulations of an } (n+2) \text{ gon}} = \sum_{k=2}^{n+1} C_{k-2}\, C_{n+3-k-2}$$

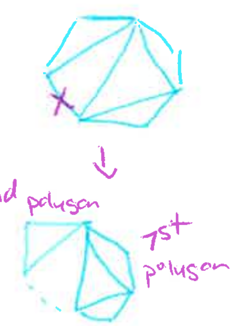
$$= \sum_{k=0}^{n-1} C_k\, C_{n-1-k}, \quad \text{with } C_0 = 1$$

1



(a 7-gon, or heptagon)

2

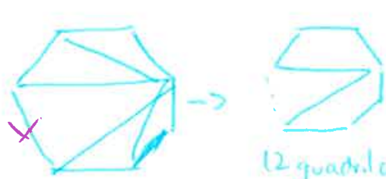


3. # of triangulations of a triangle: $C_1 = 1$

# of triangulations of a pentagon: $C_3 = 5$

But we could also have splitted it somewhere else if the initial triangulation was different.

Ex.



(2 quadrilaterals)

This is why we need to sum over all possible of sides for the 1st polygon:

$$\underset{n=5}{C_5} = \sum_{k=2}^{6} \underbrace{C_{k-2}}_{\#\text{ of triangulations of a line: } 1 = C_0} C_{5+3-k-2} = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$$

☺

- A generating function?

Using the recursion, one can deduce the generating function. For more details, you can wait until we learn generating functions next week, or read Stanley's monograph on Catalan numbers.

Theorem

The generating function for the Catalan numbers is
$$C(x) = \frac{1-\sqrt{1-4x}}{2x} =: \sum_{n\geq 0} C_n x^n. \qquad (1)$$

- A closed formula?

Equating coefficients from both sides of the equation (1) above, one get the following:

Theorem

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n}\binom{2n}{n-1}.$$

- The asymptotic behavior?

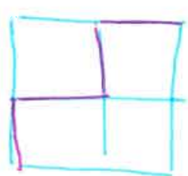
Using the generating function, one can also compute the asymptotic behavior of the Catalan numbers:
$$C_n \sim \frac{4^n}{\sqrt{\pi}\; n^{3}}.$$
(but that is a little complicated, and there are non-combinatorial symbols in it ...)

Application: Ballot sequences.

Two candidates, A and B, are running in a tight election. In the end, A will receive as many votes as B; they get n votes each. What is the probability that A is never behind B as the counting of the votes goes?

Solution: The probability is $\frac{1}{n+1} = \frac{C_n}{\binom{2n}{n}}$.

- The counting of votes is the same as writing a sequence with n A's and n B's. This explains the denominator.
- The numerator, C_n, is the number of lattice paths in an $(n \times n)$-grid that stay above the diagonal.



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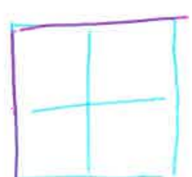

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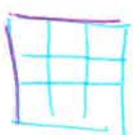

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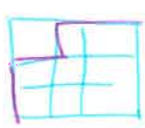

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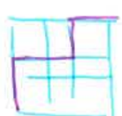

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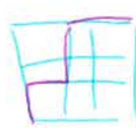

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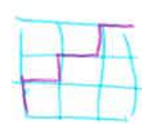

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Those paths are called Dyck paths.

To see that Dyck paths are counted by Catalan numbers, do the following:

- add one row on top of the $n \times n$ grid. You now have a $(n+1) \times n$ grid, and a path on it will be made of $(n+1)$ A's and n B's. Write the paths as words.
- We say that two words are <u>conjugate</u> if they can be obtain one from each other using a cyclic shift.

- <u>Claim 1</u>: All $2n+1$ words with $(n+1)$ A's and n B's in a conjugacy class are distinct.
- <u>Claim 2</u>: In each conjugacy class, there is exactly one path that is a Dyck path.

(If you want to see it pictured, come talk to me. It is a picture on a cylinder, so it is not good looking on paper.)

Thus, the number of Dyck paths is

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

Oh! why are the lattice paths over the diagonal the same on a $(n+1) \times n$ and on a $n \times n$ grid? To go from the first to the latter, just remove the first A.

Claim 3: it works!

<u>Reference</u>: Richard P. STANLEY. Catalan numbers, Cambridge University Press, 2015, 215 pages.