Math 68 - Algebraic Combinatorics

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Permutation modules

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Today, I hope it will be more clear how we can think of Specht modules being isomorphic to submodules of $\mathbb{C}S_n$.

## Definition

The permutation module of a partition $\lambda$ is the vector space spanned by the tabloids of shape $\lambda$. It is denoted $M^\lambda$.

## Example

$$\begin{array}{l}\hline 2\ 3\\ \hline 1\\ \hline\end{array} + \begin{array}{l}\hline 1\ 2\\ \hline 3\\ \hline\end{array} \in M^{(2,1)}$$

$$M^{(n)} = \mathbb{C}\cdot \begin{array}{l}\hline 1 2 \cdots n\\ \hline\end{array}$$

$M^{(1^n)} \cong \mathbb{C}S_n$ Example: $n=3$ $M^{(1,1,1)} = \left\langle \begin{array}{c}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}, \begin{array}{c}\hline 1\\ \hline 3\\ \hline 2\\ \hline\end{array}, \begin{array}{c}\hline 2\\ \hline 1\\ \hline 3\\ \hline\end{array}, \begin{array}{c}\hline 2\\ \hline 3\\ \hline 1\\ \hline\end{array}, \begin{array}{c}\hline 3\\ \hline 1\\ \hline 2\\ \hline\end{array}, \begin{array}{c}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} \right\rangle$.

Since polytabloids are linear combination of tabloids, $S^\lambda \subseteq M^\lambda$.

## Algebra of words

Define an alphabet $A$, and the algebra of words over $\mathbb{C}$ to be $\mathbb{C}\langle A\rangle$. The items in $\mathbb{C}\langle A\rangle$ are linear combinations of words, i.e. sequences of letters.

$\mathbb{C}A^n$ is the vector space spanned by words of length $n$.

## Theorem

$$\mathbb{C}A^n \supseteq \bigoplus_{\lambda \vdash n} M^\lambda$$

Given a tabloid, one can find a word by writing at position $i$ the letter $j$ provided that $i$ lies in the $j$-th row of the tabloid.

Example

$$\begin{array}{l} \hline 1\ 4\ 6 \\ \hline 2\ 5 \\ \hline 3 \\ \hline \end{array} \longleftrightarrow abcaba$$

That correspondance can be extended linearly.

Example

Let $t = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$.

Then, $e_t = \begin{array}{l} \hline 1\ 2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{l} \hline 2\ 3 \\ \hline 1 \\ \hline \end{array} \longleftrightarrow aab - baa$.

Here, permutations correspond to tabloids of shape $1^n$.

## Decomposing $M^\lambda$

We know that $S^\lambda \subseteq M^\lambda$. However, there are other things in $M^\lambda$ that are not in $S^\lambda$.

$S^{(2,1)} = \langle e_{\substack{12\\3}}, e_{\substack{13\\2}} \rangle$, but $\begin{array}{l} \hline 1\ 2 \\ \hline 3 \\ \hline \end{array} \notin S^{(2,1)}$. It obviously belongs to $M^{(2,1)}$.

We can still decompose $M^\lambda$ into modules isomorphic to some Specht modules.

**Definition** (Dominance order for partitions).

Suppose $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ are partitions of $n$.

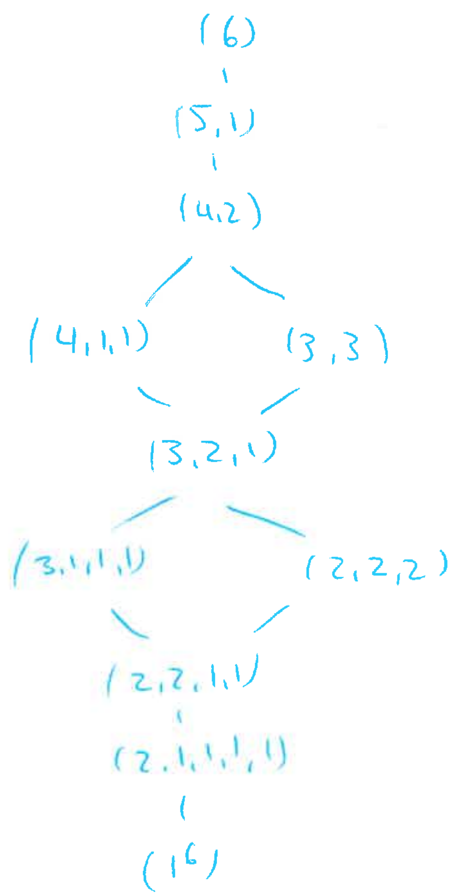
Then $\lambda$ dominates $\mu$, written $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i, \quad \text{for all } i \geq 1.$$

(We take $\lambda_i = 0$ if $i > \ell$ (and same thing for $\mu_i$, $i > m$)).

**Example**

Hasse diagram for $n = 6$




**Proposition**

Let $\theta$ be a non-zero module homomorphism from $S^\lambda$ to $M^\mu$.

Then $\lambda \trianglerighteq \mu$ and, if $\lambda = \mu$, then $\theta = c \cdot Id$, for $c \in \mathbb{C}$.

## Corollary

The permutation modules decompose as

$$M^{\mu} \cong \bigoplus_{\lambda \trianglerighteq \mu} \underbrace{m_{\lambda\mu}}_{\text{multiplicity.}} S^{\lambda}$$

The number $m_{\lambda\mu}$ is known as the kostka number, and it can be computed using Young's rule.

## Theorem (Young's rule)

The multiplicity of $S^{\lambda}$ in $M^{\mu}$ is equal to the number of SSYT of shape $\lambda$ and content $\mu$, i.e.

$$M^{\mu} \cong \bigoplus_{\lambda} m_{\lambda\mu} S^{\lambda},$$

with $m_{\lambda\mu}$; the number of SSYT of shape $\lambda$ and content $\mu$.

## Example

$$M^{(2,1)} = M^{(3)} \oplus M^{(2,1)},$$

because $\boxed{\begin{array}{|c|c|c|}1&1&2\end{array}}$ and $\begin{array}{|c|c|}\hline 1&1\\ \hline 2 \\ \cline{1-1}\end{array}$ are the only SSYT of content $(2,1)$.

## Example

Suppose $\mu = (2,2,1)$.

| $\lambda \vdash 5$ | $\lambda \trianglerighteq \mu$ | SSYT of shape $\lambda$ and content $\mu$ |
| --- | --- | --- |
| $(5)$ | ✓ | $\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\ \hline\end{array}$ |
| $(4,1)$ | ✓ | $\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2 \\ \cline{1-1}\end{array}$ |
| $(3,2)$ | ✓ | $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3 \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2 \\ \cline{1-2}\end{array}$ |
| $(3,1,1)$ | ✓ | $\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ |
| $(2,2,1)$ | ✓ | $\begin{array}{|c|c|}\hline 1&1\\ \hline 2&2 \\ \hline 3 \\ \cline{1-1}\end{array}$ |
| $(2,1,1,1)$ | ✗ | |
| $(1,1,1,1,1)$ | ✗ | |

Hence,

$$M^{(2,2,1)} \cong S^{(5)} \oplus 2S^{(4,1)} \oplus 2S^{(3,2)} \oplus S^{(3,1,1)} \oplus S^{(2,2,1)}$$

## Corollary

The symmetric group algebra decomposes as

$$\mathbb{C}S_n \cong M^{(1^n)} \cong \bigoplus_{\lambda \trianglerighteq (1^n)} m_{\lambda 1^n} S^\lambda$$

$$\cong \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda$$

**Proof**

The SSYTs of content $(1^n)$ are the SYTs.

All partitions of $n$ dominate $(1^n)$.

## Connections with the branching rule.

Some permutation modules can be expressed as induced representations.

**Example**

$$\mathbb{C}S_n \cong S^{\square} \uparrow_{S_1}^{S_2} \uparrow_{S_2}^{S_3} \cdots \uparrow_{S_{n-1}}^{S_n}$$

trivial representation

Why? This is a way to construct successively all the SYTs.

We first add 1 in the top left corner, then 2 in an outer corner of $\square$, and so on.

For example, with $n=4$:

$$\mathbb{C}S_4 \cong M^{(1,1,1,1)} \cong S^{\square} \uparrow_{S_1}^{S_2} \uparrow_{S_2}^{S_3} \uparrow_{S_3}^{S_4}$$

$$\cong \left(S^{\ydiagram{2}} \oplus S^{\ydiagram{1,1}}\right) \uparrow_{S_2}^{S_3} \uparrow_{S_3}^{S_4}$$

$$\cong \left(S^{\ydiagram{3}} \oplus 2S^{\ydiagram{2,1}} \oplus S^{\ydiagram{1,1,1}}\right) \uparrow_{S_3}^{S_4}$$

$$\cong S^{\ydiagram{4}} \oplus 3 S^{\ydiagram{3,1}} \oplus 2S^{\ydiagram{2,2}} \oplus 3 S^{\ydiagram{2,1,1}} \oplus S^{\ydiagram{1,1,1,1}}$$

SYTs of size 4

$$\ytableau{1 & 2 & 3 & 4} \quad \ytableau{1 & 2 & 3 \\ 4} \quad \ytableau{1 & 2 & 4 \\ 3} \quad \ytableau{1 & 3 & 4 \\ 2} \quad \ytableau{1 & 2 \\ 3 & 4} \quad \ytableau{1 & 3 \\ 2 & 4} \quad \ytableau{1 & 2 \\ 3 \\ 4}$$

$$\ytableau{1 & 3 \\ 2 \\ 4} \quad \ytableau{1 & 4 \\ 2 \\ 3} \quad \ytableau{1 \\ 2 \\ 3 \\ 4}$$

## Example

On page (4), we decomposed $M^{(2,2,1)}$.

It is obtained as

$$\left(S^{(2,2)} \oplus S^{(3,1)} \oplus S^{(4)}\right)\Big\uparrow_{S_4}^{S_5}$$

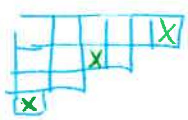
$$= 2S^{(3,2)} \oplus S^{(2,2,1)} \oplus 2S^{(4,1)} \oplus S^{(3,1,1)} \oplus S^{(5)}$$

Notice that $(2,2)$, $(3,1)$ and $(4)$ are the only shapes that dominate $(2,2)$

## Theorem (Generalized branching rule).

$$M^{(a_1,\dots,a_k)} \cong S^{(a_1)}\Big\uparrow_{S_{a_1}}^{S_{a_1+a_2}}\Big\uparrow_{S_{a_1+a_2}}^{S_{a_1+a_2+a_3}} \cdots \Big\uparrow_{S_{n-a_k}}^{S_n},$$

where $S^\lambda\uparrow_{S_i}^{S_j}$ is the direct sum of $S^\mu$ ranging over $\mu$'s such that $\mu/\lambda$ is a horizontal strip (i.e. a strip of boxes with no two of them in the same column).



x : a horizontal strip

## Example

$$M^{(2,2,1)} \cong S^{(2)}\Big\uparrow_{S_2}^{S_4}\Big\uparrow_{S_4}^{S_5}$$

$$\cong \left(S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}\right)\Big\uparrow_{S_4}^{S_5},$$

and we get what is at the top of the page.

## Corollary

All permutation modules are obtained by repeatedly inducing from the trivial representation.

Reference: Bruce E. Sagan, The Symmetric Group. §2.2, 2.4, 2.11, 4.9