Math 68 - Algebraic Combinatorics. Nadia Lafrenière

Basis for the Specht modules 11/01/2019

Recall from last class that the tableaux of shape $(2,1)$ are, with their associate polytabloids

$$t_1 = \begin{array}{cc}1&2\\3\end{array} \qquad e_{t_1} = \begin{array}{cc}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

$$t_2 = \begin{array}{cc}1&3\\2\end{array} \qquad e_{t_2} = \begin{array}{cc}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

$$t_3 = \begin{array}{cc}2&3\\1\end{array} \qquad e_{t_3} = \begin{array}{cc}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = -e_{t_2}$$

$$t_4 = \begin{array}{cc}2&1\\3\end{array} \qquad e_{t_4} = \begin{array}{cc}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = e_{t_1} - e_{t_2}$$

$$t_5 = \begin{array}{cc}3&1\\2\end{array} \qquad e_{t_5} = \begin{array}{cc}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = -e_{t_4} = e_{t_2} - e_{t_1}$$

$$t_6 = \begin{array}{cc}3&2\\1\end{array} \qquad e_{t_6} = \begin{array}{cc}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} - \begin{array}{cc}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = -e_{t_1}$$

We saw that $S^{(2,1)} = \langle e_{t_1}, e_{t_2} \rangle$ is a two-dimensional vector space.

Here, t_1 and t_2 correspond to the standard tableaux of shape $(2,1)$.

Theorem

Let S^λ be the Specht module corresponding to shape λ (i.e. the vector space spanned by polytabloids of shape λ).

A basis for S^λ is

$$\{ e_t : t \text{ is a SYT} \}.$$

The goal of this lecture is to prove that this theorem is true. We will first prove independence using posets (!), and then that it spans S^λ using a counting argument (with the Robinson-Schensted bijection).

Order on tabloids of a given shape.

Let $\{s\}$ be a tabloid.

We say $\{s\}$ has composition sequence $(\lambda^1, \lambda^2, \dots, \lambda^n)$ such that

- $\lambda^0 = (0, 0, 0, \dots, 0)$ (length = # rows in $\{s\}$)
- $\lambda^i = \lambda^{i-1} + (0, 0, \dots, 1, \dots, 0)$

  j-th position if i is in row j of $\{s\}$

Dominance order for tabloids

If $\{t\}$ has composition sequence $\mu = (\mu^1, \mu^2, \dots, \mu^n)$, then $\{s\}$ dominates $\{t\}$, denoted $\{s\} \trianglerighteq \{t\}$ if $\sum_{k=1}^{j} \lambda_k^i \geq \sum_{k=1}^{j} \mu_k^i$ for all $i \in [n]$, and for all $j \in$ [#rows of $\{s\}$]

Example

$\{s\} = \begin{array}{c} \hline 2\ 4 \\ 1\ 3 \\ \hline \end{array}$. Its composition sequence is

$(\lambda^1 = (0,1),\ \lambda^2 = (1,1),\ \lambda^3 = (1,2),\ \lambda^4 = (2,2))$

$\{s\} \trianglerighteq \begin{array}{c} \hline 3\ 4 \\ 1\ 2 \\ \hline \end{array}$ Its composition sequence is

$(\mu^1 = (0,1),\ \mu^2 = (0,2),\ \mu^{(3)} = (1,2),\ \lambda^4 = (2,2))$

$\{s\} \trianglelefteq \begin{array}{c} \hline 2\ 3 \\ 1\ 4 \\ \hline \end{array}$ Its composition sequence is

$(\mu^1 = (0,1),\ \mu^2 = (1,1),\ \mu^3 = (2,1),\ \mu^4 = (2,2))$

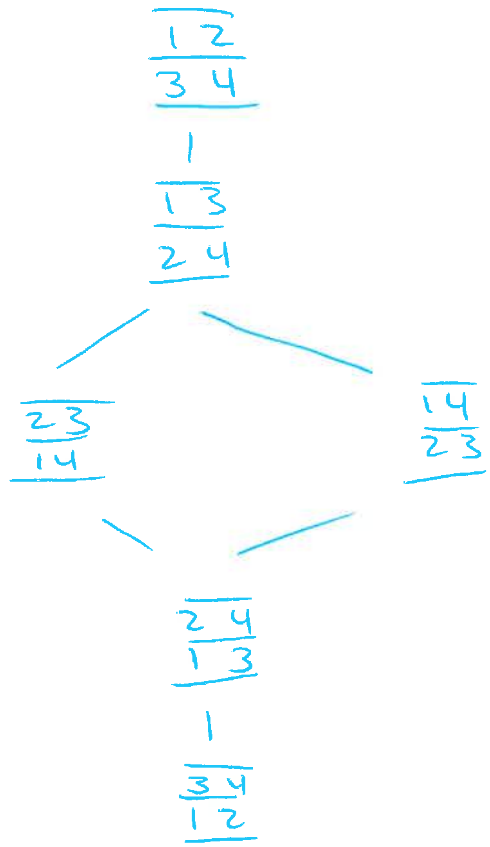
$\begin{array}{c} \hline 2\ 3 \\ 1\ 4 \\ \hline \end{array} \ \begin{matrix} \ntrianglelefteq \\ \ntrianglerighteq \end{matrix} \ \begin{array}{c} \hline 1\ 4 \\ 2\ 3 \\ \hline \end{array}$ they have composition sequences respectively

$((0,1),\ (1,1),\ (2,1),\ (2,2))$

$\quad\ \wedge \qquad\qquad\qquad\quad \vee$

$((1,0),\ (1,1),\ (1,2),\ (2,2))$.

The poset of tabloids of shape (2,2) is



The maximum is always obtained by filling the boxes from top to bottom

The minimum is obtained by filling the boxes from bottom to top

Lemma (dominance lemma for tabloids)

If $k < l$ and k appears in a lower row than l in $\{s\}$, then

$$(k,l)\{s\} \rhd \{s\}.$$

Corollary

If t is standard and $\{s\}$ appears in e_t, then $\{t\} \unrhd \{s\}$

Proof

Since $\{s\}$ appears in e_t, there exists $\sigma \in C_t$ (the column stabilizer of t) such that $\{s\} = \{\sigma t\}$. wlog, assume $s = \sigma t$.

Then, σ can be written as a product of column-preserving involutions that sends up a lower element. By the dominance lemma, $\{s\} \lhd (k,l)\{s\} \unlhd \dots \unlhd \{t\}$.

Lemma

Let $v_1, v_2, \dots, v_m$ be linear combinations of tabloids.
Suppose, for each v_i, we can choose a tabloid $\{t_i\}$ appearing in v_i such that

i. $\{t_i\}$ is maximum in v_i.

ii. the $\{t_i\}$ are all distinct.

Then $v_1, \dots, v_m$ are independent.

Proof

Choose the labels so that $\{t_1\}$ is maximal among the $\{t_i\}$, and so on for $\{t_2\}, \dots$

Conditions i. and ii. ensure that $\{t_1\}$ appears only in v_1.
Hence, if we have

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0,$$

then c_1 must be 0, because that is the only way of cancelling $\{t_1\}$.

By induction on m, we prove linear independence.

Theorem

The set $\{e_t : t \text{ is a standard tableau}\}$ is independent.

Proof

By the lemma on last page, every standard tableau t is such that $\{t\}$ is maximal among the tabloids appearing in e_t.
Using the lemma on this page, the $\{e_t : t \text{ a SYT}\}$ is independent.

We are done proving independence.
To ensure this is a basis, consider the following proposition from representation theory.

Proposition

Let G be a finite group and V_i the simple modules of $\mathbb{C}G$.

Then, $\sum_i (\dim(V_i))^2 = |G|$.

Theorem

The set $\{e_t : t \text{ a SYT}\}$ spans S^λ.

Proof

Using the Robinson-Schensted bijection, we know that

$$\sum_\lambda (f^\lambda)^2 = n!,$$

where f^λ is the number of SYT of shape λ.

Using the proposition above and the fact that the simple modules for $\mathbb{C}S_n$ are indexed by partitions, we get

$$n! = \sum_\lambda (S^\lambda)^2.$$

Since the set $\{e_t : t \text{ a SYT}\}$ is independent, we know that

$$f^\lambda \leq S^\lambda$$

for all $\lambda \vdash n$. They are equal iff $\{e_t : t \text{ a SYT}\}$ spans S^λ.

However,

$$\sum_\lambda (f^\lambda)^2 = n! = \sum_\lambda (S^\lambda)^2,$$

and so $f^\lambda = S^\lambda$, for all $\lambda \vdash n$. $\square$

Reference: Bruce E. Sagan. The Symmetric Group. § 1.10, 2.5.