Math 68 - Algebraic Combinatorics

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Specht Modules

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Goal: Understand the simple modules of $\mathbb{C}S_n$, the Specht modules.

Proposition

Let G be a finite group, and $\mathbb{C}G$ its group algebra (i.e. the vector space spanned by the elements of G).

Then, the number of simple modules of $\mathbb{C}G$ is the number of conjugacy classes of G.

Example

$G = S_n$.

The conjugacy classes of S_n are given by the permutations sharing the same cyclic type.

$$\begin{tikzcd} \text{Simple modules of } \mathbb{C}G \arrow[r, leftrightarrow] & \text{Conjugacy classes of } G \arrow[d, leftrightarrow, dashed, "\boxed{G=S_n \text{ only}}"] & \\ & \text{Cycle types} \arrow[r, leftrightarrow, dashed] & \text{Partitions} \end{tikzcd}$$

If $n=3$, then $G=S_3$.

How many simple modules of $\mathbb{C}S_3$?

- Trivial module $\left(\langle \sum_{\sigma \in S_3} \sigma \rangle \right)$ $\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$
- Sign module $\left(\langle \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \cdot \sigma \right)$ $\quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$
- Another module (called standard) of dimension 2 $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$

Young subgroup

Consider $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$.

The Young subgroup associated to λ is

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_\ell+1,\ldots,n\}}$$

$$\cong S_{\lambda_1} \times S_{\lambda_2} \times S_{\lambda_3} \times \cdots \times S_{\lambda_n}.$$

Now consider all tableaux of shape λ with content $(1,2,\ldots,n)$.

Example: $\lambda = (2,1)$

$$\begin{array}{ll} 1 & 2 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{ll} 2 & 1 \\ 3 & \end{array} \quad \begin{array}{ll} 2 & 3 \\ 1 & \end{array} \quad \begin{array}{ll} 3 & 1 \\ 2 & \end{array} \quad \begin{array}{ll} 3 & 2 \\ 1 & \end{array}$$

Most of them are not SSYTs.

Two tableaux are row-equivalent if the content of their first row is equal, of their second row is equal, ...

Example

$$\begin{array}{ll} 1 & 2 \\ 3 & \end{array} \sim \begin{array}{ll} 2 & 1 \\ 3 & \end{array} \qquad \begin{array}{c} \hline 1\;2 \\ \hline 3 \\ \hline \end{array}$$

$$\begin{array}{ll} 1 & 3 \\ 2 & \end{array} \sim \begin{array}{ll} 3 & 1 \\ 2 & \end{array} \qquad \begin{array}{c} \hline 1\;3 \\ \hline 2 \\ \hline \end{array}$$

$$\begin{array}{ll} 2 & 3 \\ 1 & \end{array} \sim \begin{array}{ll} 3 & 2 \\ 1 & \end{array} \qquad \begin{array}{c} \hline 2\;3 \\ \hline 1 \\ \hline \end{array}$$

A tabloid is a class of row-equivalent tableaux with content $1, 2, \ldots, n$

Example

With $n=3$, there is only one tabloid of shape (3) ($\overline{\underline{1\,2\,3}}$), the 3 tabloids of shape $(2,1)$ and 6 tabloids of shape $(1,1,1)$:

$$\begin{array}{c}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \quad \begin{array}{c}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}, \quad \begin{array}{c}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}, \quad \begin{array}{c}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}, \quad \begin{array}{c}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}, \quad \begin{array}{c}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}.$$

tabloids of shape λ fixed by σ

$\lambda \backslash \sigma \in S_3$	Id	(12)	(23)	(13)	(123)	(132)
(3)	1	1	1	1	1	1
(2,1)	3	1	1	1	0	0
(1,1,1)	6	0	0	0	0	0

Observations on this table.

(i) The sum of each line is $6 = \# S_3$

(ii) The sum of all the squares on a line can be divided by 6

(iii) For a same conjugacy class, all permutations have the same values on any given line.

(iv) The numbers are all (nonnegative) integers.

Definition

The row-stabilizer subgroup of a tableau t with rows $R_1, R_2, \dots, R_\ell$ is

$$R_t = S_{R_1} \times \dots \times S_{R_\ell}.$$

The column stabilizer subgroup of a tableau t with columns $C_1, \dots, C_k$ is

$$C_t = S_{C_1} \times \dots \times S_{C_k}.$$

Example

For $t = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$, $R_t = \langle (1,2), (3,4) \rangle$ and $C_t = \langle (1,3), (2,4) \rangle$.

Both are 4-element subgroups.

Tabloids are fixed under R_t (by definition).

We would like to get something similar for C_t.

Definition

Let t be a tableau and $\{t\}$ be its tabloid.

The polytabloid e_t is the linear combination

$$e_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \cdot \sigma\{t\}.$$

Example

Let $t = \begin{array}{ccc} 4 & 1 & 2 \\ 3 & 5 \end{array}$. Then, $\{t\} = \begin{array}{c} \hline 4\,1\,2 \\ \hline 3\,5 \\ \hline \end{array}$ and

$$e_t = \begin{array}{c} \hline 4\,1\,2 \\ \hline 3\,5 \\ \hline \end{array} - \begin{array}{c} \hline 3\,1\,2 \\ \hline 4\,5 \\ \hline \end{array} - \begin{array}{c} \hline 4\,5\,2 \\ \hline 3\,1 \\ \hline \end{array} + \begin{array}{c} \hline 3\,5\,2 \\ \hline 4\,1 \\ \hline \end{array}$$

$$= \begin{array}{c} \hline 1\,2\,4 \\ \hline 3\,5 \\ \hline \end{array} - \begin{array}{c} \hline 1\,2\,3 \\ \hline 4\,5 \\ \hline \end{array} - \begin{array}{c} \hline 2\,4\,5 \\ \hline 1\,3 \\ \hline \end{array} + \begin{array}{c} \hline 2\,3\,5 \\ \hline 1\,4 \\ \hline \end{array}$$

Example

Let $\lambda = (2,1)$.

Consider the three tabloids: $\left\{ \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{c} \hline 1\,3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{c} \hline 2\,3 \\ \hline 1 \\ \hline \end{array} \right\}$.

- $t_1 = \begin{array}{cc} 1 & 2 \\ 3 \end{array}$, $e_{t_1} = \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{c} \hline 3\,2 \\ \hline 1 \\ \hline \end{array} = \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{c} \hline 2\,3 \\ \hline 1 \\ \hline \end{array}$

- $t_2 = \begin{array}{cc} 1 & 3 \\ 2 \end{array}$, $e_{t_2} = \begin{array}{c} \hline 1\,3 \\ \hline 2 \\ \hline \end{array} - \begin{array}{c} \hline 2\,3 \\ \hline 1 \\ \hline \end{array}$

- $t_3 = \begin{array}{cc} 2 & 3 \\ 1 \end{array}$, $e_{t_3} = \begin{array}{c} \hline 2\,3 \\ \hline 1 \\ \hline \end{array} - \begin{array}{c} \hline 1\,3 \\ \hline 2 \\ \hline \end{array} = -e_{t_2}$.

There are 2 distinct polytabloids of shape $(2,1)$.

There also are 1 polytabloid of shape (3) and 1 of shape $(1,1,1)$.

Definition

Let $\lambda \vdash n$.

The Specht module S^λ is the module spanned by the poly-tabloids e_t, with shape$(t) = \lambda$.

Remark

The Specht modules do not lie in $\mathbb{C}S_n$, but they are isomorphic to submodules of $\mathbb{C}S_n$.

Theorem

The Specht modules $\{S^\lambda, \lambda \vdash n\}$ are a complete list of simple modules of $\mathbb{C}S_n$.

Example

Last class, I said that $\mathbb{C}S_3$ could be decomposed as a direct sum

$$\langle \sum_{\sigma \in S_n} \sigma \rangle \oplus \langle \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma \rangle \oplus 2\,\mathcal{U},$$

where $\mathcal{U}$ is some 2-dimensional module.

- $\sum_{\sigma \in S_n} \sigma$ corresponds to $\overline{\underline{1\,2\,3}}$
- $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma$ corresponds to $\begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$.
- $\mathcal{U} \cong \langle e_{\substack{12\\3}}, e_{\substack{13\\2}} \rangle$.

Reference: Bruce E. Sagan. The Symmetric Group, 2nd edition. §1.10, 2.1, 2.2, 2.3, 2.4.