Math 68 - Algebraic Combinatorics
Pólya theory: Burnside's lemma

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Goal: Enumerating inequivalent objects that are subject to a group of symmetries. This theory was developed by George Pólya and John Howard Redfield, and is hence called Pólya enumeration theory, or Pólya-Redfield theory.

Problem: Consider a row of five squares

(i) In how many ways can we color the squares into n colors?

This is n^5, since every square can be colored with any of the n colors.

(ii) Same question, but considering that two colorings are equivalent if they are one from the other after a $180°$ degree rotation?

Ex

To most coloring, we can associate exactly one other coloring by rotation, so we might think $\frac{1}{2}n^5$ is the answer, but this is not always an integer.

Some coloring are self-identical up to rotation,  is one of them, and there are n^3 such colorings (since choosing the 3 first colors is sufficient to fix the coloring).

Thus, the number of coloring is

$$\frac{1}{2}(n^5 - n^3) + n^3 = \frac{1}{2}(n^5 + n^3)$$

Is that always an integer?

The general setting for Pólya enumeration:

- A set X (like the squares above)
- A set of colors C, that might be infinite, so that a coloring is a function $f: X \to C$.
- A group G, that is a subgroup of S_X, and that acts on G by permuting the elements of X

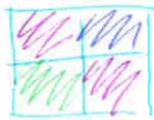
We define two colorings f and g to be equivalent (or G-equivalent) if there exists $\sigma \in G$ such that $f(x) = g(\sigma(x))$, for all $x \in X$, and we denote $f \sim g$ or $f \overset{G}{\sim} g$. (They are the same, up to a symmetry in G)

Example

Consider 4 boxes

1	2
3	4

and the coloring



given by $f(1) = \text{purple} = f(4)$, $f(2) = \text{blue}$ and $f(3) = \text{green}$.

How many equivalence classes are there under the following groups, if we are looking only at colorings with two purple boxes, one blue and one green?

(Id) The only one-element group

(V) The two-elements group containing the vertical reflexion.

(D_g) The two-elements group containing the reflexion along the main diagonal

(R) The group of all rotations

(Di) the dihedral group, i.e. the group containing all rotations and reflexions of the square.

(S) The symmetric group.

(A) The alternate group (even-length permutations)

(Id) There is no action. Choose the two purple squares, then the blue, and this fixes the green

(V) There are six classes:

(Dg) Let's start by enumerating them:

# elements in the class	2	1	1	2	2	2	2	Sum: 12

So there are 7 classes. We know there is none missing, because counting the multiplicities, we recover the number of classes in (Id), where every class contains one element.

Caveat!

As groups, V and Dg are isomorphic. However, this is not enough to have the same classes! Burnside's Lemma will be the key to why this is true.

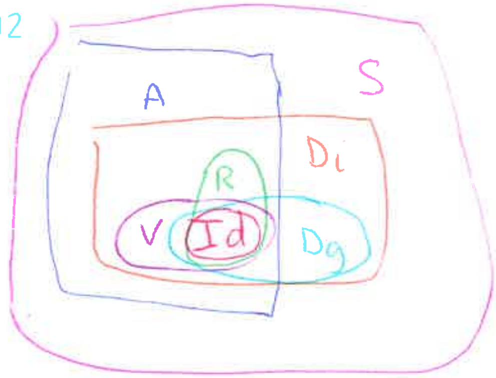
(R) There are two cases to consider: first, if the two purple are adjacent, we need to fix the first color going clockwise

Otherwise, there is only one class 

So there are three classes

(Di) We can start from the classes for R (since $R \subseteq D_i$) and meld classes. Here , since they are the same up to vertical symmetry. So there are two classes

Groups	Some elements	#G	# classes with 2 ..., 1 ... & 1 ...
Id	Id	1	$\binom{4}{2}\binom{2}{1}\binom{1}{1}=12$
V	(13)(24), Id	2	6
Dg	(123), Id	2	7
R	(1234)	4	3
Di	(1234), (23)	8	2
S	(1324)	24	1
A	(13)(24), (12)(34) (123)	12	1



(S) The Symmetric group allows one to permute all the squares, so all colorings with the same colors are in the same class.

(A) From every permutation with an even number of involutions, we can switch the two purple squares and get, in this way, all the odd permutations. So the number of classes is the same as for the symmetric group.

Fix the group and let the colors be variables

We are now going back to the original problem of fixing a group G and count the total number of inequivalent coloring. Let $x_1, x_2, x_3, x_4, \dots$ be the colors that we are allowed to take for our coloring.

The generating function F_G counts this number.

Example: Let $G = D_i$ and a set of 3 colors, x_1, x_2, x_3. Then, the generating function for the non-equivalent colorings of the four squares under G is

$$F_{D_i}(x_1, x_2, x_3) = \sum_{i+j+k=4} K(i,j,k)\, x_1^i x_2^j x_3^k$$

$$= (x_1^4 + x_2^4 + x_3^4) + (x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_1 + x_2^3 x_3 + x_3^3 x_1 + x_3^3 x_2)$$

$$+ 2(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) + 2(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2)$$

This function is a symmetric function, that is a function in which permuting the variables do not change the function. What we care about is the coefficients $K(x_1, x_2, \dots, x_n)$, since they do not change when we change the names of the colors.

Example: Can we generalize the formula in the last example to extend it to all colorings of the four squares (under the action of the dihedral group), without considering the number of colors available?

This is

$$F_{D_i}(x_1, x_2, \dots, x_n) = \sum_i x_i^4 + \sum_{i \neq j} x_i^3 x_j + 2\sum_{i<j} x_i^2 x_j^2$$

$$+ 2 \sum_{\substack{i<j \\ i \neq k \\ j \neq k}} x_i x_j x_k^2 + 3 \sum_{1 \leq i<j<k<l \leq n} x_i x_j x_k x_l .$$

The last coefficient comes from the number of choices for the color diagonal to the i-th. Here, the indices range from 1 to n, where n is the number of available colors.

The number of inequivalent colorings is thus

$$F_{D_i}(1, 1, \dots, 1) = n + 2\binom{n}{2} + 2\binom{n}{2} + 2 \cdot 3 \binom{n}{3} + 3\binom{n}{4}$$

$$= \frac{1}{8}\left(n^4 + 2n^3 + 3n^2 + 2n\right)$$ (after some computations, but they are straightforward)

Let us do some numerology...

... and rewrite

$$F_{D_i}(1, 1, \dots, 1) = \frac{1}{\# D_i} \# \{\sigma \in D_i \mid \sigma \text{ has } i \text{ cycles}\}$$

Note that the elements of D_i are

σ	$(1)(2)(3)(4)$	(1243)	$(14)(23)$	(1342)	$(13)(24)$	$(12)(34)$	$(14)(2)(3)$	(23)
# cycles	4	1	2	1	2	2	3	3

This is not a coincidence...

Lemma (Burnside's Lemma/Cauchy-Frobenius lemma)

Let X be a finite set and G be a subgroup of S_X.

For each $\sigma \in S_X$, denote

$$Fix(\sigma) = \{x \in X \mid \sigma(x) = x\},$$

the set of fixed points of σ.

The number of orbits of G is

$$\underbrace{|X/G|}_{\text{notation for the number of orbits}} = \frac{1}{|\quad|} \sum_{\sigma \in G} \# Fix(\sigma)$$

Equivalently, the number of orbits is the average number of fixed points

With that tools, we can easily compute the number of orbits in each of the groups on page 2:

G	Id	V	Dg	R	Di	S	A
$\|X/G\|$	4	2	3	1	1	1	1

For example, for A, we can split the permutations by cycle type:

Cycle types	# permutation in A_4	# fixed points
(4)	0	0
(3,1)	8	1
(2,2)	3	0
(2,1,1)	0	2
(1,1,1,1)	1	4,

So

$$|X/A_4| = \frac{1}{12}(8 \cdot 1 + 3 \cdot 0 + 1 \cdot 4) = 1$$

To prove Burnside's Lemma, recall the three following facts from grap theory:

Lemma (Lagrange's theorem (?))

Let G act on X.

(i) The distinct orbits form a partition of X.

(ii) For any $x \in X$ the stabilizer of x (i.e. $\{g \in G \mid gx = x\}$) is a subgroup of G.

(iii) If both G and X are finite, the size of the orbit of x multiplied by the size of its stabilizer gives the size of the grap:

$$|Orb(x)| \cdot |Stab(x)| = G.$$

Proof of Burnside's lemma

Starting from the right-hand side;

$$\frac{1}{|G|} \sum_{\sigma \in G} \# Fix(\sigma) = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\substack{x \in X \\ \sigma(x) = x}} 1$$

$$= \frac{1}{|G|} \sum_{x \in X} \sum_{\substack{\sigma \in G \\ \sigma(x) = x}} 1$$

$$= \frac{1}{|G|} \sum_{x \in X} |Stab_G(x)|$$

$$= \frac{1}{|G|} \sum_{x \in X} \frac{|G|}{|Orb_G(x)|}$$

$$= \sum_{x \in X} \frac{1}{|Orb_G(x)|}$$

But, using part (i) of the Lemma above, we can partition X into its orbits:

$$\Rightarrow \frac{1}{|G|} \sum_{\sigma \in G} \# Fix(\sigma) = \sum_{\substack{\mathcal{O} \text{ orbit} \\ \text{of } X \text{ under} \\ G}} \underbrace{\sum_{x \in \mathcal{O}} \frac{1}{|Orb_G(x)|}}_{1}$$

$$= \# \text{ orbits of } X \text{ under } G$$

Reference: Richard P. Stanley, Algebraic Combinatorics. §7.