Math 68 - Algebraic Combinatorics

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Matroids

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Matroids are structures that generalize vector spaces, in the way that they are (finite) sets with a notion of independance.

Definition

A (finite) matroid is a pair $(S, \mathcal{J})$, where S is a finite set and $\mathcal{J}$ is a collection of subsets of S, satisfying:

(i) $\mathcal{J}$ is nonempty, and if $J \in \mathcal{J}$ and $I \subseteq J$, then $I \in \mathcal{J}$
(order ideal property, or simplicial complex property)

(ii) If $I, J \in \mathcal{J}$ and $|I| < |J|$, there exists $x \in J \setminus I$ such that $I \cup \{x\} \in \mathcal{J}$.

The elements of $\mathcal{J}$ are the independent sets.

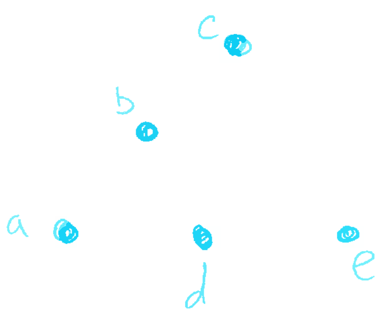
Examples

- Vector spaces
  S = a set of vectors (might not all be linearly independant)
  $\mathcal{J}$ = subsets of S containing linearly independant vectors
- Hyperplane arrangements
  Taking S as the vectors normal to the hyperplanes, and their span is a vector space.
  Moreover, if the arrangement is central, the poset we get by inclusion is isomorphic to the intersection lattice.*
  It does not mean that the same is true for the independant sets

*It might not be obvious

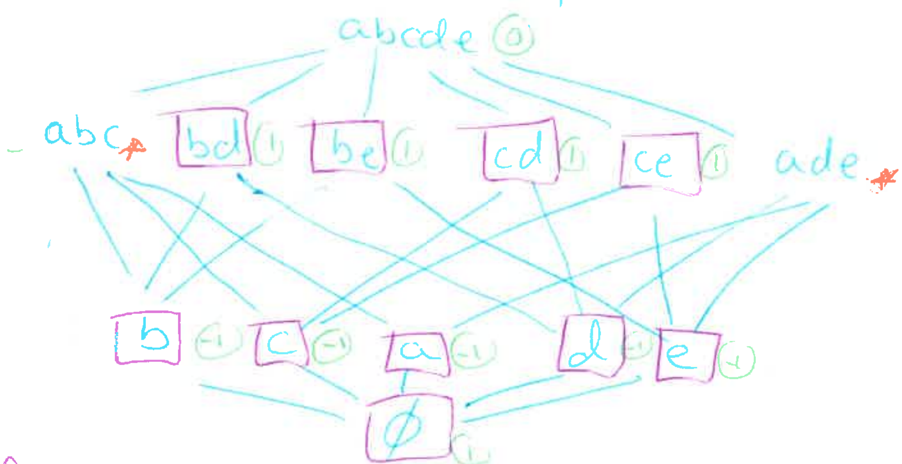
Example

Let a,b,c,d,e be points in $\mathbb{R}^2$ and consider $\{x,y\}$ to be the minimal affine subspace containing x and y (if the set has two elements, it will always be a line, but $\{x,y,z\}$ could either be a line or a plane).



$$S = \left\{ \{a,b,c,d,e\} \cap W \;\middle|\; W \text{ is an affine subspace of } \mathbb{R}^2 \right\}$$

The poset S, ordered by inclusion



are the independent sets

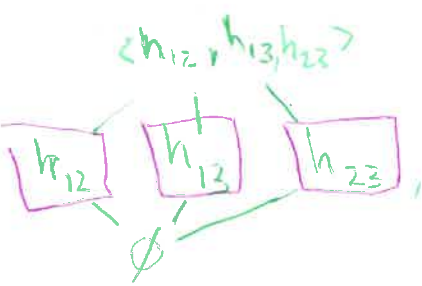
for example, $abc \notin \mathcal{I}$ because removing one independent set smaller than abc returns either ab, bc or ac, and none of them are in S.

Characterization of independence (for this poset):

A $\subseteq \{a,b,c,d,e\}$ is independant iff its rank in the poset is its cardinality.

Example

$\mathcal{B}_3$ as a hyperplane arrangement



where h_{ij} is normal to H_{ij}.

Definitions

- A basis is a maximal independent set
- A circuit is a minimal dependent set.
- The rank of $A \subseteq S$ is $\max\{|X| : X \subseteq A, X \in \mathcal{I}\}$.

Example

- For vector spaces, a basis is the usual thing we know.
- For a vector space, the rank is the dimension of the space it spans.

Example

Matroids from a graph $G = (V, E) =$

E is the set of elements. A subset of edges is independent if they do not contain a cycle (or, if they are a forest).

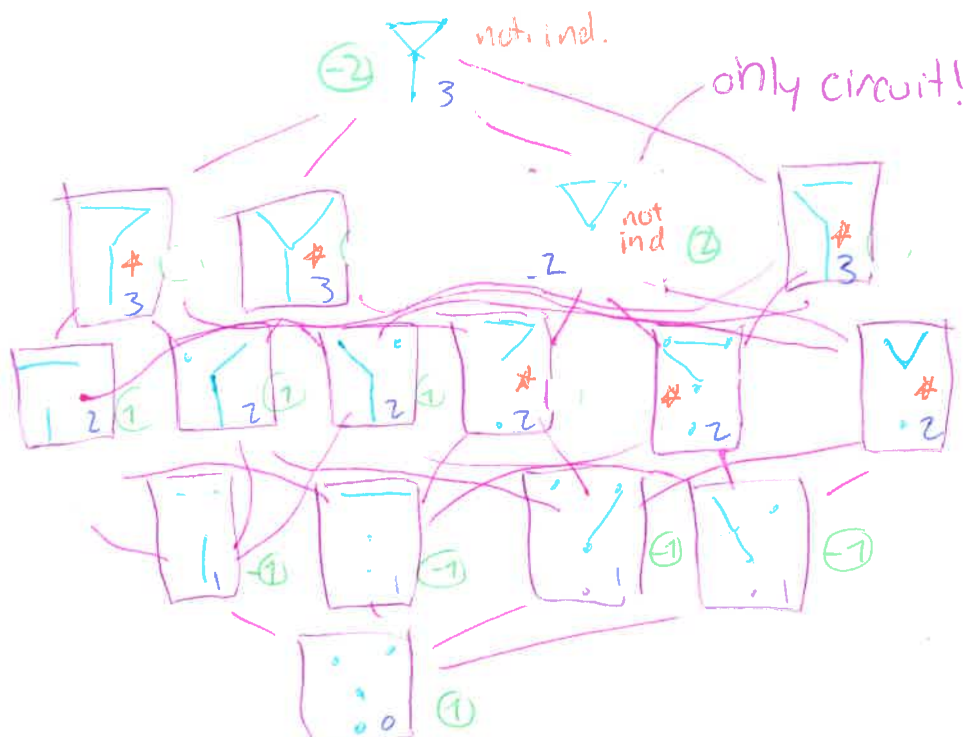
$F = \{B \subseteq E \mid B \text{ is a forest}\}$

(E, F) is a matroid

F rank

μ (for a subposet)

* not closed



Definition

Fix a total order $\triangleleft$ on S.

A <u>broken circuit</u> is a subset of S of the form $\hat{C} = C \setminus \{\min_{\triangleleft}(C)\}$, where C is a circuit

Example

For the graph on page 3, fix the following order

There is only one circuit in 2^E: $C = \triangledown$, and its minimal element is ‾. Thus, the only broken circuit of (E,F) is V.

Example

Fix $a \triangleleft b \triangleleft c \triangleleft d \triangleleft e$. S (on page 2) contains circuits abc and ade. The 2 broken circuits of S are bc and de

The <u>broken circuit complex</u> $BC(M)$ is an abstract simplicial complex (i.e. $T \in BC(M)$ and $U \subseteq T \Rightarrow U \in BC(M)$) defined by

$$BC(M) = \{T \subseteq S : T \text{ contains no broken circuit}\}$$

Example

The broken circuit complex of (E,F) is all the subsets of E that do not contain V. As a simplicial complex, it is spanned by ⊥ and ↗ (i.e. all elements of BC are included in those two) and we get the following simplicial complex.

Dimension	# faces
-1	1
0	4
1	5
2	2

Example

For the affine spaces on page 2, the broken circuits are bc and de. And all elements of S that do not contain bc nor de are included in either abd, abe, acd or ace.
Thus, the simplicial complex BC(S) is

since ab, ae, ac, ad, abd, abe, acd and ace are not in S.

Definition

The <u>closure</u> of a subset A of S for a matroid $(S,\mathcal{I})$ is

$$\operatorname{clos}(A) = \{ x \in S \mid \operatorname{rank}(A \cup \{x\}) = \operatorname{rank}(A) \}.$$

A subset A is <u>closed</u> if $\operatorname{clos}(A) = A$.
Alternatively, A is closed if it is maximal among elements of the same rank.

Definition / Theorem

The <u>characteristic polynomial</u> of a matroid $M = (S, \mathcal{I})$ admits the equivalent definitions

- $$\chi_M(t) = \sum_{X \in \operatorname{Clos}(M)} \mu(\hat{0}, X)\, t^{\operatorname{rank}(M) - \operatorname{rank}(X)},$$
  where Clos is the induced subposet of closed subsets of S, and the Möbius function is computed on that poset.
- $$\chi_M(t) = \sum_{i=0}^{\operatorname{rank}(M)} (-1)^i f_{i-1}(BC(M))\, t^{\operatorname{rank}(M) - i},$$
  where f_{i-1} is the number of faces of dimension i in $BC(M)$

Examples

We can compute in two ways the characteristic polynomials of S and (E, F).

$$\chi_S(t) = t^2 - 5t + 4 \quad \text{and} \quad \chi_S(t) = t^3 - 4t^2 + 5t - 2$$

References: Franco SALIOLA. Lecture notes for Géométrie et combinatoire.
Richard P. STANLEY. An Introduction to Hyperplane Arrangements. §3,4.