Math 68 - Algebraic Combinatorics | Nadia Lafrenière
Möbius function | 10/07/2019

Recall that, for a locally finite poset, the Möbius function of the interval $[x,z]$ is

$$\mu(x,z) = \begin{cases} 1 & \text{if } x=z \\ \sum_{x\le y<z} -\mu(x,y) & \text{otherwise.} \end{cases}$$

If a poset has a $\hat{0}$, we also write $\mu(x) = \mu(\hat{0},x)$.

Example

- Compute $\mu(x)$ for $x \in [n] = \{1,2,\dots,n\}$.

$$\mu(x) = \begin{cases} 1 & \text{if } x=1 \\ -1 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}$$

Proposition

The Möbius function of any join irreducible element x in a lattice is $\mu(x)=0$ if x is not covering $\hat{0}$.

Proof

x is join irreducible iff it covers exactly one element y.

Then,

$$\mu(x) = \sum_{z<x} -\mu(z) = \left(\sum_{z<y} -\mu(z)\right) - \mu(y) = \left(\sum_{z<y} -\mu(z)\right) - \left(\sum_{z<y} -\mu(z)\right) = 0$$

- Compute $\mu(x)$ for $x \in 2^{[n]}$ (the boolean lattice).

$$\mu(x) = (-1)^{\#x} \qquad (\text{here, } x \text{ is a subset of } 2^{[n]})$$

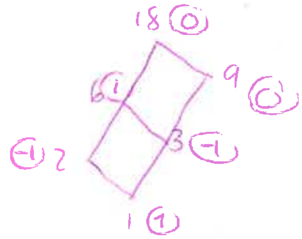
Proof

Recall that $\sum_{y\le x} \mu(y) = \delta_{\hat{0},x}$. This uniquely defines μ, so we only need to check that $\sum_{y\le x} (-1)^{\#y} = 0$ if $x \neq \hat{0}$:

$$\sum_{y\le x} (-1)^{\#y} = \sum_{k\le \#x} (-1)^k \cdot \binom{\#x}{k} = 0,$$ as you shown in the first homework

- Compute $\mu(x)$ for the divisors lattice.

(example: D_{18}, can we see a pattern?)



$$\mu(n) = \begin{cases} (-1)^m & \text{if } n \text{ is the product of } m \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

We prove only a part of it.

If n is the product of m distinct primes, we prove it by recurrence.

- If $n=1$, then $\mu(n)=1$.
- Otherwise, $\mu(n) = \sum_{d|n} \mu(d) \overset{\circledast}{=} -\sum_{k<m} \binom{m}{k}(-1)^k = -\sum_{k<m} \binom{m}{k}(-1)^k + \binom{m}{m}(-1)^m = (-1)^m$;

  where $\circledast$ is because every number is a product of k different primes, and by induction hypothesis.

Also, if n is a power of a prime number, let's say $n=p^k$, $k\geq 2$, then, p^k covers exactly one element and $\mu(p^k)$ thus vanishes.

The Möbius function appears a lot in number theory, where it has exactly the definition above. The Prime Number's Theorem can be restated as $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$ (This is not obvious!)

Theorem (Möbius Inversion Theorem).

Let P be a finite poset, and $f,g: P \to \mathbb{R}$ be two functions. Then,

(a) $f(x) = \sum_{y\geq x} g(y)$ for all $x\in P$ $\iff$ $g(x) = \sum_{y\geq x} \mu(x,y) f(y)$ for all $x \in P$.

(b) $f(x) = \sum_{y\leq x} g(y)$ for all $x\in P$ $\iff$ $g(x) = \sum_{y\leq x} \mu(y,x) f(y)$ for all $x\in P$.

Proof of (a) ($\Rightarrow$) (everything else is similar).

Assume $f(x) = \sum_{y \geq x} g(y)$ for all $x \in P$. Then,

$$\sum_{y \geq x} \mu(x,y) f(y) = \sum_{y \geq x} \mu(x,y) \sum_{z \geq y} g(z)$$

$$= \sum_{z \geq x} g(z) \sum_{x \leq y \leq z} \mu(x,y)$$

$$= \sum_{z \geq x} g(z)\, \delta_{x,z}$$

$$= g(x).$$

Applications

I- Fundamental Theorem of Difference Calculus. (FTDC)

This is a discrete analogue of calculus

Real (calculus)	Discrete
Differentiation	Forward difference
$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$	$\Delta f(n) = f(n+1) - f(n)$
Integral	Definite summation
$\int f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)}{n}$	$Sf(n) = \sum_{i=0}^{n} f(i)$

We also extend the domain of f from $\mathbb{N}$ to $\mathbb{Z}$ by $f(i) := 0$ for $i < 0$

Example

- $f(n) = n^2 \Rightarrow \Delta f(n) = (n+1)^2 - n^2 = 2n+1.$
- $g(n) = 2n+1 \Rightarrow Sg(n) = \sum_{i=0}^{n} g(i) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = \cancel{2}\,\frac{n(n+1)}{\cancel{2}} + n = n^2 + 2n$

Theorem

Given two functions $f, g : \mathbb{N} \to \mathbb{R}$,

$$f(n) = Sg(n) \text{ for all } n \geq 0 \iff g(n) = \Delta f(n-1) \text{ for all } n \geq 0.$$

Example : $f(n) = n^2+2n = Sg(n)$ when $g(n) = 2n+1$.

$\Delta f(n-1) = n^2+2n - (n-1)^2 - 2(n-1) = n^2+2n - n^2 + 2n - 1 - 2n + 1 = 2n - 1$.

Proof

We place ourselves in the setting of the poset that is the chain of order n. Here,

$$\mu(i,n) = \begin{cases} 1 & \text{if } i=n \\ -1 & \text{if } i=n-1 \\ 0 & \text{otherwise.} \end{cases}$$

The first condition of the theorem is also

$$f(n) = Sg(n) = \sum_{i=0}^{n} g(i) = \sum_{i\leq n} g(i).$$

Using Möbius inversion,

$$\begin{aligned} g(i) &= \sum_{i\leq n} \mu(i,n) f(i) \\ &= f(n) - f(n-1) \qquad \text{with the values above} \\ &= \Delta f(n-1), \end{aligned}$$

for all $n \geq 0$.

II - Principle of Inclusion and Exclusion. (PIE)

Recall that, for the boolean lattice,

$$\mu(S) = \begin{cases} 1 & \text{if } \#S \text{ is even} \\ -1 & \text{if } \#S \text{ is odd.} \end{cases}$$

Theorem (PIE)

$$\Big|S - \bigcup_{i=1}^{n} S_i\Big| = |S| - \sum_{1\leq i\leq n} |S_i| + \sum_{1\leq i<j\leq n} |S_i \cap S_j| + \dots + (-1)^m |S_1 \cap S_2 \cap \dots \cap S_m|.$$

subsets of S

The proof is a little bit involved, but it works with

$$f(I) = \Big|\bigcap_{i\in I} S_i\Big| \text{ and } g(I) = \Big|\bigcap_{i\in I} S_i - \bigcup_{j\notin I} S_j\Big| = \text{elements in all } S_i, i\in I, \text{ but in no other sets}$$

$g(I) = \sum_{J\supseteq I} \mu(I,J) f(J)$, and $I = \emptyset$. It is also Theorem 5.5.7 in [AOC].

III - Topological combinatorics

Theorem (Philip Hall's theorem)

Let P be a finite poset, and let $\hat{P}$ denote P with a $\hat{0}$ and a $\hat{1}$ adjoined. Let c_i be the number of chains of length i between $\hat{0}$ and $\hat{1}$. Then,

$$\mu_{\hat{P}}(\hat{0},\hat{1}) = \underset{\underset{0}{\shortparallel}}{c_0} - \underset{\underset{1}{\shortparallel}}{c_1} + c_2 - c_3 + c_4 - c_5 + \dots$$

Proof

$$\mu(\hat{0},\hat{1}) = \frac{1}{\zeta}(\hat{0},\hat{1}) = \frac{1}{1+(\zeta-1)}(\hat{0},\hat{1}) = (1-(\zeta-1)+(\zeta-1)^2-(\zeta-1)^3+(\zeta-1)^4-\dots)(\hat{0},\hat{1})$$

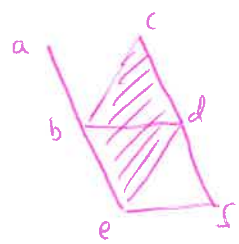
$$= c_0 - c_1 + c_2 - c_3 + c_4 - \dots$$

by result from last class!

Meaning: $\mu(\hat{0},\hat{1})$ is the Euler characteristic of a simplicial complex!

A simplicial complex on a vertex set V is a collection Δ of subsets of V (called the faces) satisfying

- $\{v\} \in \Delta \quad \forall v \in V$
- if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.



(The geometric representation)

Faces: $a, b, c, d, e, f,$ $ab, bc, bd, be,$ $cd, de, df, ef,$ $bcd, bde, \emptyset$

The dimension of the face F is $\#F-1$, and let denote the number of faces of dimension i f_i.

The Euler characteristic of Δ is $\sum_{i\geq 0}(-1)^i f_i$.

Order complex of a poset P

with vertices the elements of P and faces the chains of P.

Proposition (Philip Hall's theorem, restated).

Let P be a finite poset. Then $\mu_{\hat{P}}(\hat{0},\hat{1}) = \chi(\Delta(P)) - 1$.

↳ to account for the face of dimension -1.

References: [AOC] Bruce E. SAGAN. Combinatorics, the art of counting. §5.4, 5.5, 5.8.

[EC1] Richard P. STANLEY. Enumerative Combinatorics, volume 1. §3.8