HOMEWORK VIII

ALGEBRAIC COMBINATORICS (MATH 68)

Due November 6, 2019, at the **beginning of the class**

Collaboration among students to find key to the solution is encouraged, but each person must write the homework in his/her own words. You must write the name of the students with whom you work for each problem, as well as any written resource (web, book, etc.) that has been extensively used.
You must write the appropriate justification as part of the solutions.

(1) Consider the representation $\rho : \mathbb{C}G \to \mathrm{GL_n}(\mathbb{C})$. Its character $\chi : G \to \mathbb{C}$ is defined as $\chi(g) = \mathrm{tr}(\rho(g))$, where tr denotes the trace. Prove that the character is a class function, i.e. that it is constant over the elements of G in the same conjugacy class.

(2) Prove that each row in the table on page 3 of Wednesday's handouts are characters, for example by giving the matrices for the representation.

(3) There is an isomorphism between modules and representations, and that preserves the dimension (i.e. the dimension for a module is the *degree* of a representation). Knowing that, give the characters of the two one-dimensional representations of S_3.

(4) *(Inner product of characters)* Let χ and ψ be characters of G. Then, we define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}).$$

Also,

- if χ and ψ are characters for irreducible representations of G, $\langle \chi, \psi \rangle = \delta_{\chi,\psi}$.
- if C_1 and C_2 are conjugacy classes, then $\sum_{i<j} \chi^i \overline{\chi^j} = \frac{|G|}{|C_1|}\delta_{C_1,C_2}$, where $\bar{\cdot}$ denotes the complex conjugate, and the sum is over all irreducible representations.

(a) Find which of the representations on page 3 of Wednesday's handouts are irreducible.
(b) Without looking in an external source (but you can talk to each other), find the table of characters of S_3.
(c) Give as much information as you can on the character table of S_4. You are allowed to write only once the column for each class (since you proved in number 1 that characters are class functions.)

Good luck!