HOMEWORK VI

ALGEBRAIC COMBINATORICS (MATH 68)

Due October 23, 2019, at the **beginning of the class**

Collaboration among students to find key to the solution is encouraged, but each person must write the homework in his/her own words. You must write the name of the students with whom you work for each problem, as well as any written resource (web, book, etc.) that has been extensively used.
You must write the appropriate justification as part of the solutions.

(1) Find the number of different 4-bead, r-color necklaces, if necklaces are considered the same when one is a rotation or reflection of the other (so under the dihedral group).
(2) Ten balls are stacked in a triangular array with 1 atop 2 atop 3 atop 4 (Think of billiards). The triangular array is free to rotate in two dimensions (like on a billiard table).
   (a) Find the generating function of the number of colorings using the 10 colors $r_1, \ldots, r_{10}$.
   (b) How many colorings are there with 6 red balls and 4 blue?
(3) (a) How many distinct cubes are there under rotation if the edges are colored from a set with r colors?
   (b) How many distinct cubes are there under rotation if the vertices are colored from a set with r colors?
(4) The Rubik's cube is a famous puzzle that involves a lot of symmetries. It is made of a big cube divided into 27 smaller cubes (one is inside, with no visible face; 6 faces cubes have one visible face; 12 edges cubes have two visible faces; and 8 vertices cubes have 3 visible faces). The operations on the Rubik's cube always send the 8 vertices cubes on vertices, and the 12 edge cubes on edges. It fixes the other cubes.
   (a) Consider only the placement of the smaller cubes (and not their orientations). To what group is isomorphic the group that acts on the bigger cube? (If you don't know, tell me what are the possibilities, and what groups are not possible).
   (b) It is known that there are 2^{11} possible orientations for the 12 edges, and 3^7 orientations for the 8 vertices. How many possible configurations of the Rubik's Cube are possible? (In other words, how big is the orbit of the Rubik's cube?)
   (c) If you accidentally break your Rubik's Cube into the 27 smaller cubes, and you try to rebuild it without really paying attention (but still placing vertices onto vertices, edges onto edges, and so on), what are the odds that the cube you made is equivalent (under the Rubik's cube group action) to the original Rubik's cube?

Good luck!