Pólya Counting (lecture 1/2)

Usually coins have two sides, heads and tails, but if you want to cheat at coin flipping, you'll want 3 coins:

HT, HH, TT.

Importantly, TH is the "same" as HT, so there are 3, not 4, coins.

This is because the group $\mathbb{Z}_2$ acts on the set of coins, and its orbits are $\{HT, TH\}$, $\{HH\}$, and $\{TT\}$.

Pólya Counting generalizes this idea to any situation where a group acts on the objects of interest.

<u>Example</u>: Suppose we color the four corners of a square with the colors red and blue. How many different colorings are there if we allow the square to move around?

So, for example, we consider the colorings

$$\begin{array}{ccc} R & - & B \\ | & & | \\ B & - & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & - & B \\ | & & | \\ R & - & B \end{array}$$

to be <u>equivalent</u>.

The first step is to figure out which group is acting on these colorings.

This group is D_4, the symmetries of the square.

To be precise, label the vertices as

$$\begin{array}{ccc} 1 & - & 2 \\ | & & | \\ 4 & - & 3 \end{array}$$

Then rotating clockwise by $90°$ is

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

and flipping about the line 13 is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

These two elements generate

$$D_4 = \{e, \rho, \rho^2, \rho^3, \tau, \tau\rho, \tau\rho^2, \tau\rho^3\}.$$

Let $X = \{\text{all colorings of the square}\}$. For all $x \in X$ and $g \in G$,

x is equivalent to gx

Therefore, to count inequivalent colorings, we want to count <u>orbits</u>. The orbit of $x \in X$ is

$$\mathrm{orb}(x) = \{gx : g \in G\}.$$

<u>Orbit-Counting Lemma</u> (aka Burnside's lemma): Suppose the group G acts on the set X. Then

$$\#\text{orbits} = \text{average } \# \text{ fixed points} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}(g)|,$$

where $\mathrm{fix}(g) = \{x \in X : gx = x\}$.

To prove this, we introduce a bit more notation and prove a lemma.

Suppose G acts on the set X.

The <u>stabilizer</u>, $\mathrm{stab}(x)$ of $x \in X$ is defined as
$$\mathrm{stab}(x) = \{g \in G : gx = x\}.$$
Note that $\mathrm{stab}(x)$ is a subgroup of G.

<u>Orbit-Stabilizer Theorem:</u>

The map $g\,\mathrm{stab}(x) \mapsto gx$ is a bijection between $G/\mathrm{stab}(x)$ and $\mathrm{orb}(x)$.

<u>Proof</u>: First we must show the map is well-defined. Suppose
$$g\,\mathrm{stab}(x) = h\,\mathrm{stab}(x),$$
so $g = hs$ for some $s \in \mathrm{stab}(x)$. Then we have
$$gx = hsx = hx,$$
so the map is well-defined.

Clearly the map is onto: if $y \in \mathrm{orb}(x)$, then $y = gx$ for some $g \in G$, so y is the image of $g\,\mathrm{stab}(x)$.

Now suppose $gx = hx$. Then $g^{-1}hx = x$, so $g^{-1}h \in \mathrm{stab}(x)$, and thus $g\,\mathrm{stab}(x) = h\,\mathrm{stab}(x)$. ■

<u>Proof of the Orbit-Counting Lemma:</u>

Consider
$$\sum_{g \in G} |\mathrm{fix}(g)| = |\{(g,x) \in G \times X : gx = x\}|$$
$$= \sum_{x \in X} |\mathrm{stab}(x)|.$$
By the Orbit-Stabilizer Theorem,
$$|G/\mathrm{stab}(x)| = |\mathrm{orb}(x)|,$$
so using Lagrange's Theorem,
$$\frac{|G|}{|\mathrm{stab}(x)|} = |\mathrm{orb}(x)|,$$
and thus
$$|\mathrm{stab}(x)| = \frac{|G|}{|\mathrm{orb}(x)|}.$$

Making this substitution yields
$$\sum_{g \in G} |\mathrm{fix}(g)| = \sum_{x \in X} \frac{|G|}{|\mathrm{orb}(x)|}$$
$$= |G| \sum_{x \in X} \frac{1}{|\mathrm{orb}(x)|}.$$
Now note that each orbit A occurs $|A|$ times in this sum, so the sum is the # of orbits:
$$\sum_{x \in X} \frac{1}{|\mathrm{orb}(x)|} = \sum_{\substack{\text{orbits}\\ A}} \sum_{x \in A} \frac{1}{|A|}$$
$$= \sum_{\substack{\text{orbits}\\ A}} \frac{|A|}{|A|}$$
$$= \#\,\text{orbits}$$
Therefore
$$\sum_{g \in G} |\mathrm{fix}(g)| = |G|\,(\#\,\text{orbits}). \quad ■$$

Returning to the colored squares example, we see that we need to count fixed "points", i.e., fixed colorings.

$g \in G$	$g \cdot \begin{smallmatrix}1&2\\4&3\end{smallmatrix}$	$\lvert fix(g) \rvert$
e	$\begin{smallmatrix}1&2\\4&3\end{smallmatrix}$	16
σ	$\begin{smallmatrix}4&1\\3&2\end{smallmatrix}$	2
σ^2	$\begin{smallmatrix}3&4\\2&1\end{smallmatrix}$	4
σ^3	$\begin{smallmatrix}2&3\\1&4\end{smallmatrix}$	2
τ	$\begin{smallmatrix}1&4\\2&3\end{smallmatrix}$	8
$\tau\rho$	$\tau \begin{smallmatrix}4&1\\3&2\end{smallmatrix} = \begin{smallmatrix}4&3\\1&2\end{smallmatrix}$	4
$\tau\rho^2$	$\tau \begin{smallmatrix}3&4\\2&1\end{smallmatrix} = \begin{smallmatrix}3&2\\4&1\end{smallmatrix}$	8
$\tau\rho^3$	$\tau \begin{smallmatrix}2&3\\1&4\end{smallmatrix} = \begin{smallmatrix}2&3\\1&4\end{smallmatrix}$	4
		48

Therefore the number of inequivalent colorings is $48/8 = 6$.