Review of last lecture

An *independent set* in a graph is a subset $I \subseteq V$ of vertices, no two adjacent.

A *maximal independent set (MIS)* is an independent set that is not contained in another.

In 1965, Erdős asked: How many MISes can a graph on n vertices have?

Moon-Moser Theorem: The maximum number of MISes in a graph with n vertices is

$$g(n) = \begin{cases} 3^k & \text{if } n = 3k, \\ 4 \cdot 3^{k-1} & \text{if } n = 3k+1, \\ 2 \cdot 3^k & \text{if } n = 3k+2. \end{cases}$$

Union Proposition: For all graphs G and H we have

$$m(G \cup H) = m(G)\,m(H).$$

Neighborhood Proposition: For any vertex $v \in G$,

$$m(G) \le m(G-v) + m(G - N[v]),$$

where

$$N[v] = \{v\} \cup \{u : u \sim v\}.$$

Proof of Moon-Moser: Our proof is by induction on n. The theorem is easy to check for $n \le 5$, so we assume $n \ge 6$. Let G be a graph on n vertices.

Case 0: If G contains a vertex v of degree 0, then clearly

$$m(G) = m(G-v) \le g(n-1) < g(n).$$

Case 1: If G contains a vertex v of degree 1 then suppose $v \sim w$. Then by the Neighborhood Proposition,

$$\begin{aligned} m(G) &\le m(G-w) + m(G - N[w]) \\ &\le 2g(n-2) \quad \text{[write out]} \\ &\le g(n), \end{aligned}$$

with equality iff $n = 3k+1$ or $3k+2$.

Case ≥3: If G contains a vertex v of degree ≥ 3, then

$$\begin{aligned} m(G) &\le m(G-v) + m(G-N[v]) \\ &\le g(n-1) + g(n-4) \\ &\le g(n), \end{aligned}$$

with equality iff $n = 3k+1$.

Case 2: If *every* vertex of G has degree 2, then G is the disjoint union of a set of cycles. If all of these cycles have 3 vertices, then $n = 3k$ and $m(G) = g(n)$, and we are done.

Thus we may assume that at least one of these cycles has $\ell \geq 4$ vertices.

By our Union Proposition,

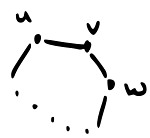
$$m(G) \leq m(C_\ell)\, g(n-\ell),$$

so it suffices to show that

$$m(C_\ell) < g(\ell)$$

for $\ell \geq 4$.

Label its vertices



We then have

$$\begin{aligned} m(C_\ell) &\leq m(C_\ell - w) + m(C_\ell - N[w]) \\ &\leq m(C_\ell - w - u) \\ &\quad + m(C_\ell - w - N[u]) \\ &\quad + m(C_\ell - N[w]) \\ &\leq 2g(\ell-3) + g(\ell-4) \\ &< g(\ell) = 3g(\ell-3). \quad \blacksquare \end{aligned}$$

Integer Complexity

The complexity of the integer n is the least number of 1s needed to represent it using only

- addition,
- multiplication, and
- parentheses.

For example:

$6 =$	$(1+1)(1+1+1)$	5
$7 =$	" $+1$	6
$8 =$	$(1+1)(1+1)(1+1)$	6
$9 =$	$(1+1+1)(1+1+1)$	6
$10 =$	" $+1$	7
	or $(1+1)(1+1+1+1+1)$	
	or $(1+1)((1+1)(1+1)+1)$.	
$11 =$	$\cdots$	8
$12 =$	$(1+1+1)(1+1+1+1)$, or...	7

This notion was introduced in 1953 by Mahler and Popken.

Note that if $c(n)$ denotes the complexity of n, then

$$c(n) = \min_{\substack{d \mid n \\ e \in [n-1]}} \left\{ c(d) + c\left(\tfrac{n}{d}\right),\ c(e) + c(n-e) \right\}.$$

This, however, is not very nice.

Questions abound:

1. For prime p, does $c(p) = 1 + c(p-1)$?
2. Is $c(n) \sim (3+\varepsilon)\log_3 n$?
3. What's the largest m with $c(m) = n$?

① and ② are still open.
Selfridge (1980s) answered ③ with an inductive proof.
We give a proof using the Moon-Moser Theorem.

Suppose that m can be expressed with n 1's, and write out this expression.

Now replace the 1s by $x_1, \dots, x_n$ (first 1 becomes x_1, last becomes x_n). If we expand this polynomial, we get <u>m terms</u>.

Now define the graph G with vertices $[n]$ by

$i \sim j$ iff x_i & x_j occur together in one of the terms.

This graph has $m \le g(n)$ maximal cliques. ■