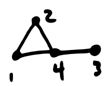
Graphs

Formal definition: a graph is a set V of vertices equipped with a set E of size 2 subsets of V called edges.

Ex: $V = [4]$
$E = \{\{1,2\},\{1,4\},\{2,4\},\{3,4\}\}$



We write $i \sim j$ if $\{i,j\} \in E$.

Sometimes we...

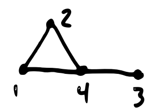
- allow loops,
- direct the edges,
- allow multiple edges between two vertices,
- put weights on the edges.

But usually we study simple graphs.

An independent set in a graph is a subset $I \subseteq V$ of vertices, no two adjacent.

A clique is a subset $C \subseteq V$ of vertices, where there is a edge connecting every pair of vertices.

Ex:



independent set: $\{2,3\}$, $\{1,3\}$
clique: $\{1,2,4\}$, $\{3,4\}$.

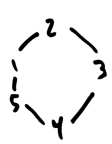
Ramsey's Theorem: Fix k. Every sufficiently large graph contains either a clique or an independent set with at least k vertices.

An independent set is maximal (an MIS) if it is not properly contained in another independent set.

Ex: Let C_n denote the cycle on the vertices $[n]$ where

$$i \sim j \iff i = j \pm 1 \pmod{n}$$

C_5:



How many MISes does C_n have?

Define $a_n = \#\text{MISes in } C_n$, so:

$$a_2 = 2$$
$$a_3 = 3$$
$$a_4 = 2$$

Proposition: For $n \geq 5$, $a_n = a_{n-2} + a_{n-3}$.

Proof: Consider an MIS $I \subseteq C_n$. Since $n \geq 5$, $|I| \geq 2$. Let $i < j$ denote the greatest two elements of I.

Clearly $j-i$ is either 2 or 3. If $j-i = 2$ then removing j gives an MIS of C_{n-2}. If $j-i=3$, then removing j gives an MIS of C_{n-3}.

Inverting this map is easy. For every MIS $I \subseteq C_{n-2}$, let i denote it greatest entry and add $i+2$. For $I \subseteq C_{n-3}$, add $i+3$. $\square$

These are the Perrin numbers.

Perrin's Theorem: If p is prime, then $p \mid a_p$.

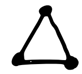
Proof: Midterm.

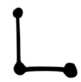
The least non-prime n such that $n \mid a_n$ was found in 1982; it is $521^2 = 271441$.

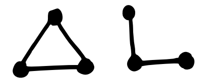
Erdős in 1965: How many MISes can a graph on n vertices have?

Define

$$m(G) = \#\text{MISes in } G$$
$$g(n) = \max\{m(G) : G \text{ has } n \text{ vertices}\}.$$

Ex: $G =$ , $m(G) = 3$.

$H =$ , $m(H) = 2$.

$G \cup H =$  $m(G \cup H) = 6$.

Proposition: For all graphs G and H, we have

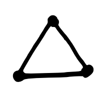
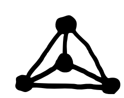
$$m(G \cup H) = m(G)\, m(H).$$

Proof: For any MIS $I \subseteq G \cup H$, $I \cap G$ must be an MIS of G, and $I \cap H$ must be an MIS of H. Conversely, if I is an MIS of G and J is an MIS of H then $I \cup J$ is an MIS of $G \cup H$. $\square$

Conjecture: For all $n \geq 2$,

$$g(n) = \begin{cases} 3^k & n = 3k \\ 4 \cdot 3^{k-1} & n = 3k+1 \\ 2 \cdot 3^k & n = 3k+2. \end{cases}$$

Achieved by disjoint unions of

, , and 

Moon-Moser: This is correct.

First, we need a technical lemma.

For any vertex v of G, its <u>(open) neighborhood</u> is

$$N(v) = \{ u : u \sim v \}.$$

Its closed neighborhood is

$$N[v] = \{u\} \cup N(u).$$

<u>Proposition</u>: For any vertex $v \in G$,

$$m(G) \leq m(G-v) + m(G-N[v]).$$

<u>Proof</u>: Take an MIS $I \subseteq G$. If $v \in I$, then $I - v$ is an MIS of $G - N[v]$, and conversely, if J is an MIS of $G - N[v]$, then $J \cup \{v\}$ is a MIS of G. Every MIS I of G which doesn't contain v is also an MIS of $G - v$. $\square$